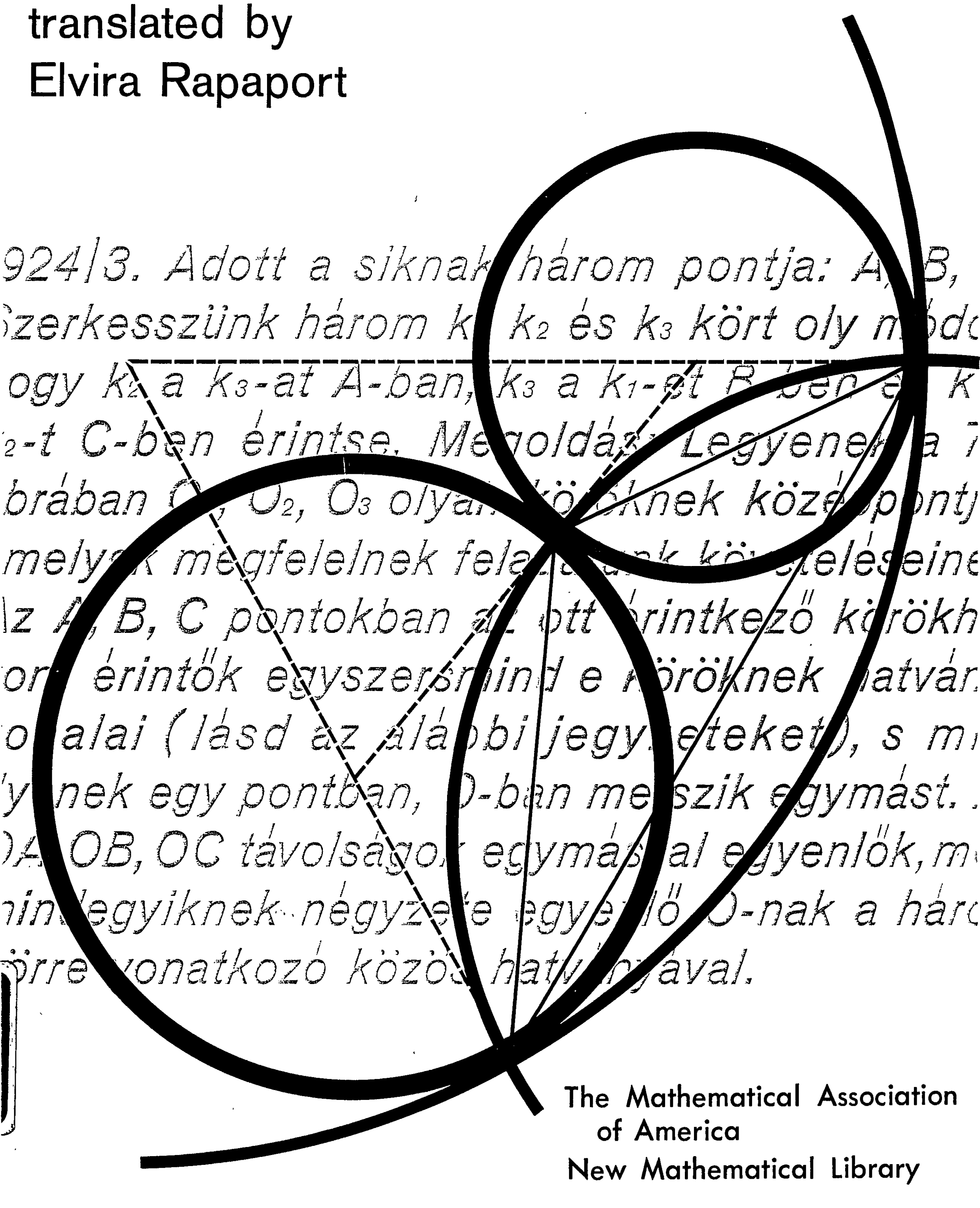
HUNGARIAN PROBLEM BOOK II

based on the
Eötvös Competitions, 1906-1928
translated by
Elvira Rapaport



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HUNGARIAN PROBLEM BOOK

BASED ON THE EÖTVÖS COMPETITIONS, 1906–1928

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HUNGARIAN PROBLEM BOOK

BASED ON THE EÖTVÖS COMPETITIONS, 1906–1928

REVISED AND EDITED BY

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The best way to learn mathematics is to *do* mathematics, and each book includes problems, some of which may require considerable thought. The reader is urged to acquire the habit of reading with paper and pencil in hand; in this way mathematics will become increasingly meaningful to him.

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HUNGARIAN PROBLEM BOOK

BASED ON THE EÖTVÖS COMPETITIONS, 1906–1928

From the Hungarian Prefaces

The first of these contests was held in 1894 by the Mathematical and Physical Society (of Hungary) in honor of its founder and president, the distinguished physicist Baron Loránd Eötvös, who became minister of education that year. To commemorate the event, the contests are given every fall and are open to high school graduates of that year. The contestants work in classrooms under supervision; the Society selects the two best papers, and the awards—a first and second Eötvös Prize—are given to the winners by the president himself at the next session of the Society.

The present volume, appearing on the tenth anniversary of Eötvös' death, contains the contests held to date. While it utilizes the winners' work, the solutions are in general not those found by the students ....

The names of the winners are listed; their papers appeared in full in *Matematikai és Fizikai Lapok*, the Journal of the Society; here, however, the solutions were changed to suit the didactic aim of the book.

Some of my notes give definitions and proofs of theorems used in the solutions. Others serve to point out the connection between problems and famous results in literature. In some instances I was able to give a glimpse of the essence of an entire subject matter; in others the mere statement of a general theorem had to suffice ....

There are few prerequisites. A person who has learned to solve quadratic equations and knows plane geometry can solve many of the problems. If he also knows trigonometry, he can solve most of them. So little of the material taught in the last two years of high school (in Hungary) is needed here that a younger student can easily learn it from books.

However, this book is meant not only for students and teachers. Anyone who retained an interest in mathematics in his adult life can find things of note and value here and will be gratified to see how much can be achieved with the elementary material to which high schools must restrict themselves.

How should the reader use this book? All I can say is: without frenzy. With a serious interest and perseverence, everyone will find the way best suited *to him* in order to benefit from the varied material contained in it . . . .

József Kürschák

Budapest, April 9th, 1929

Problems of the Mathematics Contests, edited by József Kürschák, was published originally in 1929. The first edition was quickly sold out and the Ministry of Education commissioned a new edition. We are undertaking this work with pleasure and in the hope of contributing to the attainment of Kürschák's goal. This volume will soon be followed by another containing a similar treatment of the contests held since.

The new edition required certain changes. By and large, Kürschák's approach was retained, especially his notes which were meant to widen the reader's horizon. We augmented these notes here and there, changing them only to conform to present-day high school curricula. For example, we added permutations, the binomial theorem, half-angle formulas, etc. We added a few new notes. We included some new solutions that seemed strikingly simple or ingenious, and so we deviated from the contestants' work more than the first edition did . . . .

We made a few technical changes to facilitate handling. Thus, we repeated the problems before giving the solution and we unified the notation, thereby necessitating alterations in and additions to the figures . . . .

We sincerely hope to have come up to expectation and that this work will give pleasure and profit to many.

György Hajós
Gyula Neukomm
János Surányi

Budapest, September, 1955

Preface to the American Edition

In recent times much effort has been devoted to the improvement of mathematics teaching on all levels. Thus it is only natural that we search for further stimulants and improvements in this direction, as well as for new means of discovering and developing the dormant abilities which may exist in our society. The Monograph Project of the School Mathematics Study Group has such a purpose. The present translation of the Hungarian problem collection of the Eötvös Competition serves this goal. Since I am one of the few still existing links between the present mathematical generation and an older one that witnessed the first phase of the interesting development of this competition, I was asked to write a few introductory words to the English version of this collection.

The Eötvös Competition was organized in Hungary in 1894 and played a remarkable role in the development of mathematics in that small country.† The competition was open to all freshmen entering the university; the publication of the problems and the names of the winners was from the beginning a public event of first class interest. Among the winners during the period of the first decade of the competition were such men as Fejér, von Kármán, Haar, Riesz, and numerous others who later became internationally known. With some short interruptions due to wars and related conditions the competition has been carried on to the present day, though the name was changed, and the organization and scope of the competition have become much broader in recent years. The essence however has remained the same. The problems are almost all

† Before the first world war Hungary had 19 million inhabitants; at present it has about 10 million.

from high school material (no calculus is included), they are of an elementary character, but rather difficult, and their solution requires a certain degree of insight and creative ability. Any amount of aid in the form of books or notes is permitted.

Mathematics is a human activity almost as diverse as the human mind itself. Therefore it seems impossible to design absolutely certain and effective means and methods for the stimulation of mathematics on a large scale. While the competitive idea seems to be a powerful stimulant, it is interesting to observe that it was and is still almost completely absent from academic life in Germany although mathematics has flourished in that country throughout the last two hundred years. The organization of the Eötvös Competition in Hungary was probably suggested by British and French examples that had existed in those countries for a long time. We mention in particular the "Mathematical Tripos" in Cambridge, England and the "Concours" examination problems for admission to the "Grandes Écoles" in France. These early examples suggest also that some sort of preparation is essential to arouse public interest, to attract the best competitors and to give them proper recognition. In England the participation in the Tripos is preceded by systematic coaching, and in France the public schools offer facilities to prepare for the "Concours" examinations. In Hungary a similar objective was achieved by a Journal published primarily for high school students as another natural stimulant to the student's preparation for participation in the competition upon entering the university.†

The Journal was organized almost simultaneously with the competition, i.e. in 1894, by Dániel Arany; for many years it was edited by the able high school teacher László Rácz‡ and later by various other teachers of high quality. The articles were supplied partly by teachers and partly by mathematicians affiliated with the university, mostly younger persons. The Journal carried articles primarily from elementary mathematics, much triangle geometry, some projective and descriptive geometry, algebra and occasionally some number theory, later also some ventures into calculus. But the most important and most fertile part was the

† A good account of the Eötvös Competition and of the Journal is given in an article by Tibor Radó: "On mathematical life in Hungary", *American Mathematical Monthly*, vol. 39 (1932), pp. 85–90. (One slight correction has to be made on p. 87, line 6: There *was* a girl winner, first prize, 1908.)

‡ His name will go down in history for a second reason: Rácz was the teacher of J. von Neumann in high school. Cf. the Obituary Note by S. Ulam, *Bulletin of the American Mathematical Society*, vol. 64 (1958), pp. 1–49; on p. 2 the name Rácz appears in distorted spelling.

problem section; it occupied a large part of the content and was essentially written for the students and by the students. The best solution sent in was printed with the name and school of the author, and a list of the others who sent in correct solutions was given.

I remember vividly the time when I participated in this phase of the Journal (in the years between 1908 and 1912); I would wait eagerly for the arrival of the monthly issue and my first concern was to look at the problem section, almost breathlessly, and to start grappling with the problems without delay. The names of the others who were in the same business were quickly known to me and frequently I read with considerable envy how they had succeeded with some problems which I could not handle with complete success, or how they had found a better solution (that is, simpler, more elegant or wittier) than the one I had sent in. The following story may not be accurate in all details but it is certainly revealing:

"The time is about 1940, the scene is one of the infamous labor camps of fascist Hungary just at the beginning of its pathetic transformation from semi-dictatorship to the cannibalism of the Nazi pattern. These camps were populated mostly by Jewish youth forced to carry out some perfectly useless tasks. One young man (at present one of the leading mathematicians of Hungary) was in the camp; let us call him Mr. X. He was panting under the load of a heavy beam when the sergeant shouted at him in a not too complimentary manner, addressing him by his last name. The supervising officer stood nearby, just a few steps away, and said: 'Say, did I hear right, your name is X?' 'Yes,' was the answer. 'Are you by chance the same X who worked years ago in the High School Journal?' 'Yes,' was again the answer. 'You know, you solved more, and more difficult problems than any one of us and we were very envious of you.' The end of the story is that Mr. X received more lenient treatment in the camp and later even had some mathematical contact with the all-powerful officer."

The profound interest which these young men took in the Journal was decisive in many of their lives. The intensive preoccupation with interesting problems of simple and elementary character and the effort of finding clear and complete answers gave them a new experience, the taste of creative intellectual adventure. Thus they were bound finally and unalterably to the jealous mistress that mathematics is. There remained still the question of what special studies to undertake, whether it should be mathematics or physics or engineering; but this was after all a secondary matter; the main road was charted for life. We may think

of the adage of Kronecker who compares mathematicians with lotus eaters: "Wer einmal von dieser Kost etwas zu sich genommen hat, kann nie mehr davon lassen." (He who has once tasted of this fruit can never more forswear it.)

And a final observation. We should not forget that the solution of any worth-while problem very rarely comes to us easily and without hard work; it is rather the result of intellectual effort of days or weeks or months. Why should the young mind be willing to make this supreme effort? The explanation is probably the instinctive preference for certain values, that is, the attitude which rates intellectual effort and spiritual achievement higher than material advantage. Such a valuation can only be the result of a long cultural development of environment and public spirit which is difficult to accelerate by governmental aid or even by more intensive training in mathematics. The most effective means may consist of transmitting to the young mind the beauty of intellectual work and the feeling of satisfaction following a great and successful mental effort. The hope is justified that the present book might aid exactly in this respect and that it represents a good step in the right direction.

Gábor Szegő

Stanford University, February, 1961

Problems

1906 Competition

1906/1. Prove that, if $\tan(\alpha/2)$ is rational (or else, if α is an odd multiple of π so that $\tan(\alpha/2)$ is not defined), then $\cos\alpha$ and $\sin\alpha$ are rational; and, conversely, if $\cos\alpha$ and $\sin\alpha$ are rational, then $\tan(\alpha/2)$ is rational unless α is an odd multiple of π so that $\tan(\alpha/2)$ is not defined.

1906/2. Let K, L, M, N designate the centers of the squares erected on the four sides (outside) of a rhombus. Prove that the polygon $KLMN$ is a square.

1906/3. Let $a_1, a_2, a_3, \cdots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, 3, \cdots, n$. Prove that, if n is odd, the product

$$(a_1 - 1)(a_2 - 2)(a_3 - 3)\cdots(a_n - n)$$

is an even number.

1907 Competition

1907/1. If p and q are odd integers, prove that the equation

$$x^2 + 2px + 2q = 0 \tag{1}$$

has no rational roots.

1907/2. Let P be any point inside the parallelogram $ABCD$ and let R be the radius of the circle through A, B, and C. Show that the distance from P to the nearest vertex is not greater than R.

1907/3. Let

$$\frac{r}{s} = 0\,.\,k_1k_2k_3\cdots$$

be the decimal expansion of a rational number.† Prove that at least two of the numbers

$$\sigma_1 = 10\frac{r}{s} - k_1, \qquad \sigma_2 = 10^2\frac{r}{s} - (10k_1 + k_2),$$

$$\sigma_3 = 10^3\frac{r}{s} - (10^2k_1 + 10k_2 + k_3), \quad \cdots$$

are equal.

1908 Competition

1908/1. Given two odd integers a and b; prove that $a^3 - b^3$ is divisible by 2^n if and only if $a - b$ is divisible by 2^n.

1908/2. Let n be an integer greater than 2. Prove that the nth power of the length of the hypotenuse of a right triangle is greater than the sum of the nth powers of the lengths of the legs.

1908/3. A regular polygon of 10 sides (a regular decagon) may be inscribed in a circle in the following two distinct ways: Divide the circumference into 10 equal arcs and (1) join each division point to the next by straight line segments, (2) join each division point to the next but two by straight line segments. (See Figures 48 and 49.) Prove that the difference in the side lengths of these two decagons is equal to the radius of their circumscribed circle.

† If this is a terminating decimal, all k_i from a certain one on are 0.

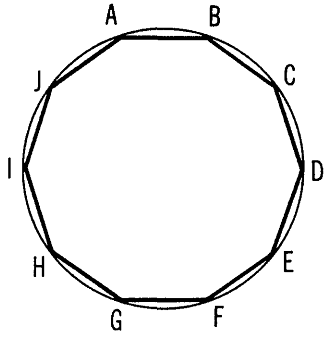

Figure 48

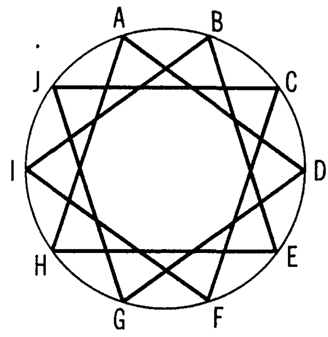

Figure 49

1909 Competition

1909/1. Consider any three consecutive natural numbers. Prove that the cube of the largest cannot be the sum of the cubes of the other two.

1909/2. Show that the radian measure of an acute angle is less than the arithmetic mean of its sine and its tangent.

1909/3. Let A_1, B_1, C_1 be the feet of the altitudes of $\triangle ABC$ drawn from the vertices A, B, C respectively, and let M be the orthocenter (point of intersection of altitudes) of $\triangle ABC$. Assume that the orthic triangle† $A_1B_1C_1$ exists. Prove that each of the points M, A, B, and C is the center of a circle tangent to all three sides (extended if necessary) of $\triangle A_1B_1C_1$. What is the difference in the behavior of acute and obtuse triangles ABC?

1910 Competition

1910/1. If a, b, c are real numbers such that

$$a^2 + b^2 + c^2 = 1$$

prove the inequalities

$$-\tfrac{1}{2} \leq ab + bc + ca \leq 1.$$

† The triangle whose vertices are the feet of the altitudes of the original triangle.

1910/2. Let a, b, c, d and u be integers such that each of the numbers

$$ac, \quad bc + ad, \quad bd$$

is a multiple of u. Show that bc and ad are multiples of u.

1910/3. The lengths of sides CB and CA of $\triangle ABC$ are a and b, and the angle between them is $\gamma = 120°$. Express the length of the bisector of γ in terms of a and b.

1911 Competition

1911/1. Show that, if the real numbers a, b, c, A, B, C satisfy

$$aC - 2bB + cA = 0 \quad \text{and} \quad ac - b^2 > 0,$$

then

$$AC - B^2 \leq 0.$$

1911/2. Let Q be any point on a circle and let $P_1P_2P_3 \cdots P_8$ be a regular inscribed octagon. Prove that the sum of the fourth powers of the distances from Q to the diameters P_1P_5, P_2P_6, P_3P_7, P_4P_8 is independent of the position of Q.

1911/3. Prove that $3^n + 1$ is not divisible by 2^n for any integer $n > 1$.

1912 Competition

1912/1. How many positive integers of n digits exist such that each digit is 1, 2, or 3? How many of these contain all three of the digits 1, 2, and 3 at least once?

1912/2. Prove that for every positive integer n, the number

$$A_n = 5^n + 2 \cdot 3^{n-1} + 1$$

is a multiple of 8.

1912/3. Prove that the diagonals of a quadrilateral are perpendicular if and only if the sum of the squares of one pair of opposite sides equals that of the other.

1913 Competition

1913/1. Prove that for every integer $n > 2$

$$(1 \cdot 2 \cdot 3 \cdots n)^2 > n^n.$$

1913/2. Let O and O' designate two diagonally opposite vertices of a cube. Bisect those edges of the cube that contain neither of the points O and O'. Prove that these midpoints of edges lie in a plane and form the vertices of a regular hexagon.

1913/3. Let d denote the greatest common divisor of the natural numbers a and b, and let d' denote the greatest common divisor of the natural numbers a' and b'. Prove that dd' is the greatest common divisor of the four numbers

$$aa', \quad ab', \quad ba', \quad bb'.$$

1914 Competition

1914/1. Let A and B be points on a circle k. Suppose that an arc k' of another circle, l, connects A with B and divides the area inside the circle k into two equal parts. Prove that arc k' is longer than the diameter of k.

1915/2. Triangle ABC lies entirely inside a polygon. Prove that the perimeter of triangle ABC is not greater than that of the polygon.

1915/3. Prove that a triangle inscribed in a parallelogram has at most half the area of the parallelogram.

1916 Competition

1916/1. If a and b are positive numbers, prove that the equation

$$\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x+b} = 0$$

has two real roots, one between $a/3$ and $2a/3$, and one between $-2b/3$ and $-b/3$.

1916/2. Let the bisector of the angle at C of triangle ABC intersect side AB in point D. Show that the segment CD is shorter than the geometric mean† of the sides CA and CB.

1916/3. Divide the numbers

1, 2, 3, 4, 5

into two arbitrarily chosen sets. Prove that one of the sets contains two numbers and their difference.

$$(2) \qquad y^2 - xy + x^2 - b = 0$$

are rational, prove that the solutions are integers.

1917/2. In the square of an integer a, the tens' digit is 7. What is the units' digit of a^2?

1917/3. Let A and B be two points inside a given circle k. Prove that there exist (infinitely many) circles through A and B which lie entirely in k.

1918 Competition

1918/1. Let AC be the longer of the two diagonals of the parallelogram $ABCD$. Drop perpendiculars from C to AB and AD extended. If E and F are the feet of these perpendiculars, prove that

$$AB \cdot AE + AD \cdot AF = (AC)^2.$$

1918/2. Find three distinct natural numbers such that the sum of their reciprocals is an integer.

1918/3. If $a, b, c;\ p, q, r$ are real numbers such that, for every real number x,

$$ax^2 + 2bx + c \geq 0 \qquad \text{and} \qquad px^2 + 2qx + r \geq 0,$$

1922/2. Prove that

$$x^4 + 2x^2 + 2x + 2$$

is not the product of two polynomials

$$x^2 + ax + b \qquad \text{and} \qquad x^2 + cx + d$$

in which a, b, c, d are integers.

1922/3. Show that, if $a, b, \cdots, n$ are distinct natural numbers, none divisible by any primes greater than 3, then

$$\frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{n} < 3.$$

1923 Competition

1923/1. Three circles through the point O and of radius r intersect pairwise in the additional points A, B, C. Prove that the circle through the points $A, B,$ and C also has radius r.

1923/2. If

$$s_n = 1 + q + q^2 + \cdots + q^n,$$

and

$$S_n = 1 + \frac{1+q}{2} + \left(\frac{1+q}{2}\right)^2 + \cdots + \left(\frac{1+q}{2}\right)^n,$$

prove that

$$\binom{n+1}{1} + \binom{n+1}{2} s_1 + \binom{n+1}{3} s_2 + \cdots + \binom{n+1}{n+1} s_n = 2^n S_n.$$

1923/3. Prove that, if the terms of an infinite arithmetic progression of natural numbers are not all equal, they cannot all be primes.

1924 Competition

1924/1. Let a, b, c be fixed natural numbers. Suppose that, for every positive integer n, there is a triangle whose sides have lengths a^n, b^n, and c^n respectively. Prove that these triangles are isosceles.

1924/2. If O is a given point, l a given line, and a a given positive number, find the locus of points P for which the sum of the distances from P to O and from P to l is a.

1924/3. Let A, B, and C be three given points in the plane; construct three circles, k_1, k_2, and k_3, such that k_2 and k_3 have a common tangent at A, k_3 and k_1 at B, and k_1 and k_2 at C.

1925 Competition

1925/1. Let a, b, c, d be four integers. Prove that the product of the six differences

$$b - a, \quad c - a, \quad d - a, \quad d - c, \quad d - b, \quad c - b$$

is divisible by 12.

1925/2. How many zeros are there at the end of the number

$$1000! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 999 \cdot 1000\,?$$

1925/3. Let r be the radius of the inscribed circle of a right triangle ABC. Show that r is less than half of either leg and less than one fourth of the hypotenuse.

1926 Competition

1926/1. Prove that, if a and b are given integers, the system of equations

$$x + y + 2z + 2t = a$$

$$2x - 2y + z - t = b$$

has a solution in integers x, y, z, t.

1926/2. Prove that the product of four consecutive natural numbers cannot be the square of an integer.

1926/3. The circle k' rolls along the inside of circle k; the radius of k is twice the radius of k'. Describe the path of a point on k'.

1927 Competition

1927/1. Let the integers a, b, c, d be relatively prime† to

$$m = ad - bc.$$

Prove that the pairs of integers (x, y) for which $ax + by$ is a multiple of m are identical with those for which $cx + dy$ is a multiple of m.

1927/2. Find the sum of all distinct four-digit numbers that contain only the digits 1, 2, 3, 4, 5, each at most once.

1927/3. Consider the four circles tangent to all three lines containing the sides of a triangle ABC; let k and k_c be those tangent to side AB between A and B. Prove that the geometric mean of the radii of k and k_c does not exceed half the length of AB.

1928 Competition

1928/1. Prove that, among the positive numbers

$$a, \quad 2a, \quad \cdots, \quad (n-1)a,$$

there is one that differs from an integer by at most $1/n$.

1928/2. Put the numbers $1, 2, 3, \cdots, n$ on the circumference of a circle in such a way that the difference between neighboring numbers is at most 2. Prove that there is just one solution (if regard is paid only to the order in which the numbers are arranged).

1928/3. Let l be a given line, A and B given points of the plane. Choose a point P on l so that the longer of the segments AP, BP is as short as possible. (If $AP = BP$, either segment may be taken as the "longer" one.)

† Two integers are called *relatively prime* if they have no common divisor greater than 1.

Solutions

1906 Competition

1906/1. Prove that, if $\tan(\alpha/2)$ is rational (or else, if α is an odd multiple of π so that $\tan(\alpha/2)$ is not defined), then $\cos\alpha$ and $\sin\alpha$ are rational; and, conversely, if $\cos\alpha$ and $\sin\alpha$ are rational, then $\tan(\alpha/2)$ is rational unless α is an odd multiple of π so that $\tan(\alpha/2)$ is not defined

First Solution. The identities

(1)
$$\sin\alpha = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}$$

(2)
$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$$

(3)
$$1 = \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}$$

yield

$$\sin\alpha = \frac{2\cos(\alpha/2)\sin(\alpha/2)}{\cos^2(\alpha/2)+\sin^2(\alpha/2)}, \qquad \cos\alpha = \frac{\cos^2(\alpha/2)-\sin^2(\alpha/2)}{\cos^2(\alpha/2)+\sin^2(\alpha/2)}.$$

If $\tan(\alpha/2)$ is rational, that is, $\tan(\alpha/2) = m/n$, where m, n are integers, $n \neq 0$, then $\tan(\alpha/2)$ is certainly defined, $\alpha \neq (2k+1)\pi$, and $\cos(\alpha/2) \neq 0$, so that we may divide numerators and denominators by $\cos^2(\alpha/2)$ and obtain

$$(4) \qquad \sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}, \qquad \cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}.$$

When m/n is substituted for $\tan(\alpha/2)$ in (4) and when the fractions are simplified, they take the form

$$\sin\alpha = \frac{2mn}{m^2+n^2}, \qquad \cos\alpha = \frac{n^2-m^2}{m^2+n^2},$$

where numerators and denominators are integers, so that $\sin\alpha$ and $\cos\alpha$ are rational.

If $\alpha = (2k+1)\pi$, $\tan(\alpha/2)$ is not defined, $\cos(\alpha/2) = 0$, and $\sin(\alpha/2) = \pm 1$. In that case (1) and (2) yield the rational numbers

$$\sin\alpha = 0, \qquad \cos\alpha = -1.$$

To prove the converse, we observe that the sum of identities (2) and (3) is

$$(5) \qquad 1 + \cos\alpha = 2\cos^2\frac{\alpha}{2}.$$

If $\sin\alpha$, $\cos\alpha$ are rational and if α is not an odd multiple of π, i.e. $\cos^2(\alpha/2) \neq 0$, then we may divide identity (1) by (5) obtaining the rational number

$$\frac{\sin\alpha}{1+\cos\alpha} = \tan\frac{\alpha}{2},$$

since the quotient of two rational numbers is a rational number.

Second Solution. Represent $\sin\alpha$ and $\cos\alpha$ in the usual way by means of the coordinates x and y of the point P on the unit circle (Figure 40); then $\cos\alpha = x$, $\sin\alpha = y$. Connect the point A: $(-1, 0)$ with P if these points are not identical, that is, if $\alpha \neq (2k+1)\pi$. Angle $PAO = \alpha/2$ since it is inscribed while α is the corresponding central angle of the unit circle. Moreover, since P lies on the semicircle with diameter AB, lines AP and BP are perpendicular. Therefore,

denoting the angle between BP and the x-axis by β and recalling that the slope of a straight line is the tangent of the angle it makes with the x-axis, and that slopes of perpendicular lines are negative reciprocals of each other, we have

$$\tan\beta = -\frac{1}{\tan(\alpha/2)}.$$

This relation tells us that if $\tan(\alpha/2)$ is rational then lines AP and BP have rational slopes. Moreover, since one passes through the point A with rational coordinates and the other through point B with rational coordinates, it is clear that their point of intersection P will have rational coordinates $x = \cos\alpha$, $y = \sin\alpha$.†

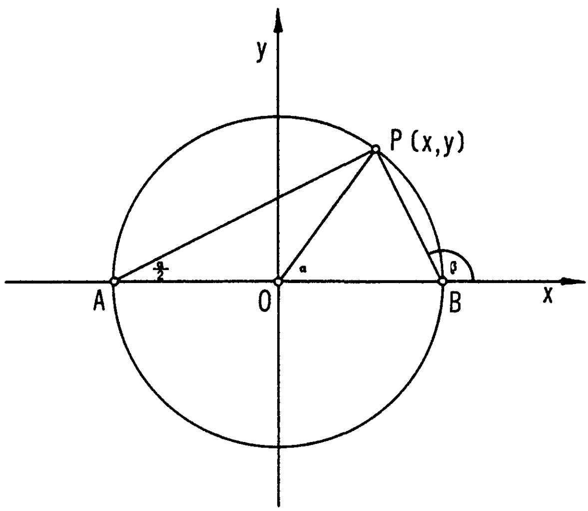

Figure 40

If $\tan(\alpha/2)$ is not defined, then α is an odd multiple of π and P coincides with A. In this case its coordinates are

$$x = \cos\alpha = -1 \quad \text{and} \quad y = \sin\alpha = 0,$$

again rational.

† This conclusion is based on the fact that addition, subtraction, multiplication and division of rational numbers yield rational numbers, and the calculation of the intersection of two straight lines involves only these operations. (The calculation of the slope of a line when two points on the line are known also involves only these operations.)

Conversely, if $\cos\alpha$, $\sin\alpha$ are rational, then P has rational coordinates, and if $\alpha \neq (2k+1)\pi$, then the slope of PA is rational since it passes through two points whose coordinates are rational.†

1906/2. Let K, L, M, N designate the centers of the squares erected on the four sides (outside) of a rhombus. Prove that the polygon $KLMN$ is a square.

First Solution. The configuration consisting of the rhombus and the four squares is symmetric with respect to each diagonal of the rhombus. Since these diagonals are perpendicular and since points K, L, M, N are not on these lines, polygon $KLMN$ is a rectangle whose center is the point O at which the diagonals of the rhombus intersect; see Fig. 41.

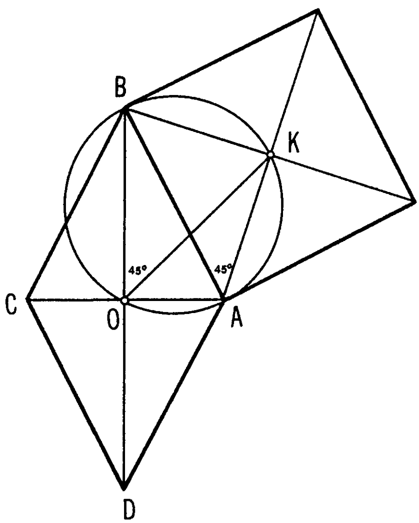

Figure 41

In order to show that this rectangle is a square, we show that its diagonals intersect at right angles or, equivalently, that the segment KO makes a 45° angle with either of the diagonals of the rhombus, say with the segment BO.

Since $\measuredangle AOB = \measuredangle AKB = 90°$, quadrilateral $AKBO$ may be inscribed in the circle with diameter AB. Since K is the center of a square, $KB = KA$ so that $\measuredangle BAK = 45°$. Hence $\measuredangle BOK$ which also intercepts arc BK is also 45°.

† See previous footnote.

Second Solution. The above theorem is true not only for a rhombus, but also for an arbitrary parallelogram $ABCD$. In order to prove this generalization, take, for example, the square belonging to side BC of the parallelogram and construct on each side of it a parallelogram congruent to $ABCD$ (see Fig. 42). The two segments issuing from a vertex of the square (outward) are equal and perpendicular to each other. Use them as the sides of a square to be constructed at each of the four vertices. The resulting figure is invariant under a rotation through 90° about the point L. Such a rotation carries the center K of the square over AB into the center M of the square over CD so that ΔKLM is an isosceles right triangle. If this triangle is rotated by 180° about the center of parallelogram $ABCD$, it will be carried into ΔMNK. Therefore $KLMN$ is a square.

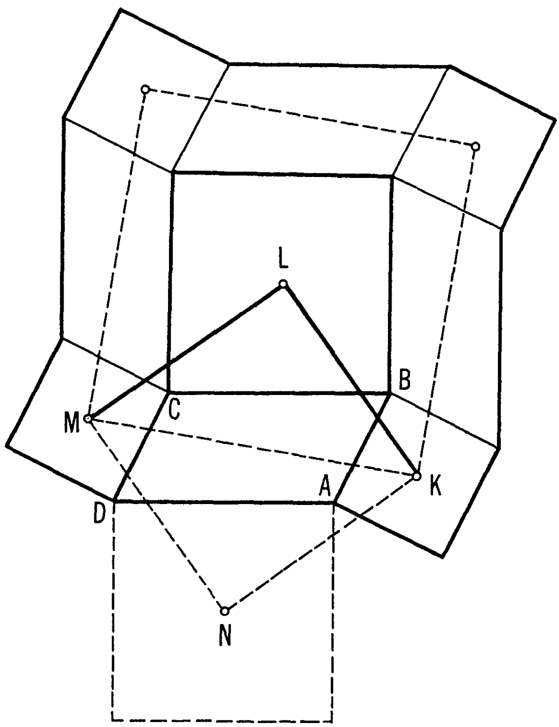

Figure 42

1906/3. Let $a_1, a_2, a_3, \cdots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, 3, \cdots, n$. Prove that, if n is odd, the product

$$(a_1 - 1)(a_2 - 2)(a_3 - 3)\cdots(a_n - n)$$

is an even number.

First Solution. We shall prove that at least one factor in this product is even. Since n is odd, we may write $n \doteq 2k + 1$, where k is an integer, and observe that our product has $2k + 1$ factors. Moreover, of the numbers $1, 2, 3, \cdots, 2k + 1$, exactly $k + 1$ are odd because

$$1 = 2 \cdot 1 - 1,$$

$$3 = 2 \cdot 2 - 1,$$

$$5 = 2 \cdot 3 - 1,$$

$$\cdots\cdots\cdots$$

$$2k + 1 = 2(k + 1) - 1.$$

Since the a's also consist of the numbers from 1 to $2k + 1$, exactly $k + 1$ of the a's are odd. Therefore, there are exactly $2(k + 1) = n + 1$ odd numbers among the $2n$ numbers $a_1, a_2, \cdots, a_n$; $1, 2, \cdots, n$ appearing in the above product. However, there are only n factors. Hence, at least one of the factors contains two odd numbers, say a_m and m so that $a_m - m$ is even. Therefore the entire product is divisible by 2.

Note. The pigeonhole principle. The above solution is based on the useful principle contained in the following simple example: If there are more than three objects to be accommodated in only three boxes, at least one of these boxes must contain more than one object. More abstractly: If more than n objects are divided into n classes, then at least one class contains more than one object.

Second Solution. The number of factors is odd and their sum is clearly 0, which is even. This would be impossible if all the factors were odd. Consequently at least one of the factors is even, and so is their product.

1907 Competition

1907/1. If p and q are odd integers, prove that the equation

$$x^2 + 2px + 2q = 0 \tag{1}$$

has no rational roots.

Solution. a) *No solution of eq.* (1) *is an odd integer.* For, if x is odd, x^2 is also odd while $2px + 2q$ is even. The sum $x^2 + 2px + 2q$ is therefore odd and not zero.

b) *If p and q are odd, no solution of eq.* (1) *is an even integer.* For, if x is even, $x^2 + 2px$ is a multiple of 4 while $2q$ is not. The sum $(x^2 + 2px) + 2q$ is not divisible by 4, and hence not equal to zero.

c) *If p and q are odd, no root of eq.* (1) *is rational.* We write eq. (1) in the form

$$(x + p)^2 = p^2 - 2q,$$

and assume x is a rational fraction. Since by a) and b) x is not an integer, we now exclude this possibility. But then $x + p$ and $(x + p)^2$ are rational, but not integers, while $p^2 - 2q$ is an integer.

Note. Equations with integral coefficients. The last part of the above proof was based on the fact that the square of a non-integral rational fraction is not an integer. This is a special case of the theorem: *If the coefficients of the algebraic equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

are integers, then every rational root of this equation is an integer.

Suppose

$$x = \frac{r}{s}$$

is a rational root of the above equation. (By definition a rational number may be written as a ratio of integers r and s, $s \neq 0$.) We may assume that r and s have no common factor, because we can always achieve this by writing the fraction in lowest terms.

We now substitute r/s in the equation and multiply both sides by s^n obtaining

$$r^n + a_1r^{n-1}s + a_2r^{n-2}s^2 + \cdots + a_{n-1}rs^{n-1} + a_ns^n = 0$$

or

$$r^n = -s(a_1r^{n-1} + a_2r^{n-2}s + \cdots + a_{n-1}rs^{n-2} + a_ns^{n-1}).$$

If s were greater than 1 in absolute value, it would be a product of positive primes (cf. 1896/1, Note). In other words, s would be divisible by some prime, and therefore r^n would have to be divisible by that same prime. According to 1894/1, Note to the Second Solution, r would then also be divisible by that prime so that r and s would have a common divisor, contrary to our assumption that r/s was in lowest terms. It follows that $s = 1$; hence r/s is an integer.

In particular, if $n = 2$ and $a_1 = 0$, our general equation takes the form

$$x^2 + a_2 = 0.$$

By our theorem, if a_2 is an integer and x is rational, then x is an integer;

or, equivalently, if x is not an integer but if x is rational, then a^2 is not an integer. Hence the square of a non-integral rational number is not an integer.

The theorem (and its proof) can easily be generalized as follows:† *If the coefficients of the algebraic equation*

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

are integers, and if there is a rational root r/s in lowest terms, then r is a divisor of a_n and s is a divisor of a_0.

1907/2. Let P be any point inside the parallelogram $ABCD$ and let R be the radius of the circle through A, B, and C. Show that the distance from P to the nearest vertex is not greater than R.

First Solution. In view of the symmetry in a parallelogram, we may assume that point P is inside or on the boundary of $\triangle ABC$. For, if P were inside $\triangle ADC$, a circle of radius R could be circumscribed about $\triangle ADC$ which is congruent to $\triangle ABC$, the labels B and D could be interchanged, and the problem would not be altered.

In order to show that at least one of the distances PA, PB, PC is not greater than R we shall prove the theorem: *The distance from a point P inside or on the boundary of a triangle to the nearest vertex is not greater than the radius of the circumscribed circle.*

The proof is based on the following lemma: *If P is a point inside or on the boundary of a right triangle, then its distance from either endpoint of the hypotenuse does not exceed the length of the hypotenuse.*

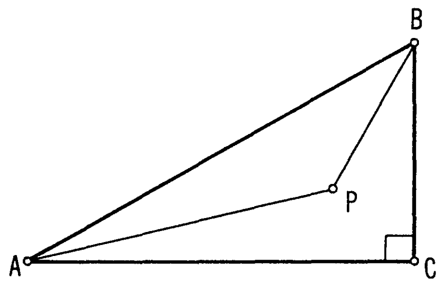

Figure 43

This is obvious when P is on the hypotenuse. When P is inside the right triangle ABC (Figure 43) or on one of its legs, then the largest

† See, for example, I. Niven, *Numbers: Rational and Irrational*, p. 58, Theorem 4.3, in this series.

angle of $\triangle ABP$ is at P. Therefore the. side AB, opposite P, is longer than AP or BP.

With the aid of this lemma we now prove the theorem. Let A, B, C designate the vertices of the triangle and O the center of its circumcircle (Figures 44, 45). Denote by A', B', and C' the points at which perpendiculars from O meet the sides opposite A, B, and C respectively. A point P inside or on the boundary of $\triangle ABC$ is inside or on the boundary of at least one of the triangles AOB', $B'OC$, COA', $A'OB$, BOC', $C'OA$. Suppose AOB' contains P; then the lemma gives

$$AP \leq AO = R.$$

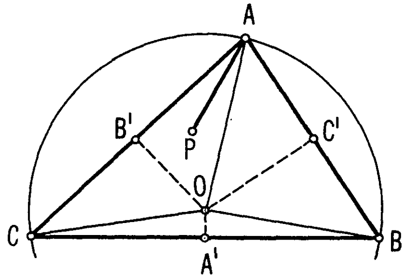

Figure 44

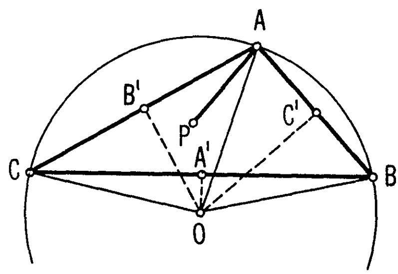

Figure 45

Second Solution. We shall now give a different proof of the theorem used in the first solution. This second proof is based on this fact: *If P is inside or on the boundary of $\triangle ABC$, then any straight line not through P divides the plane into two half planes, one containing P and the other not; the half plane containing P contains also at least one of the vertices of $\triangle ABC$.*

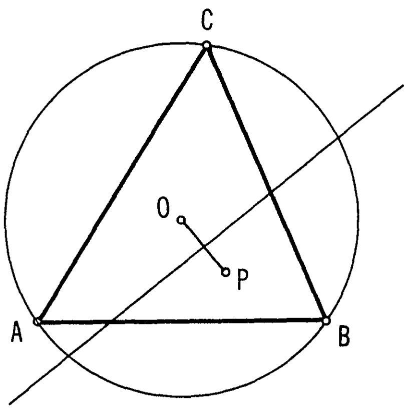

Figure 46

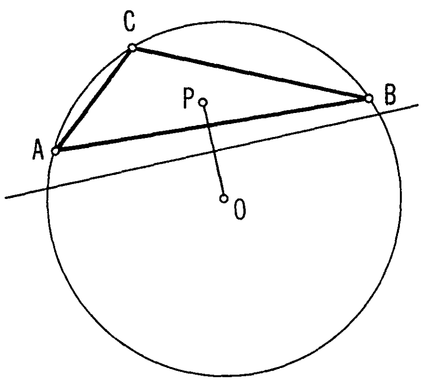

Figure 47

Now if P is the center O of the circumcircle, the theorem used for the first solution is clearly true. If $P \neq O$, construct the perpendicular bisector of the segment PO (Figures 46, 47). Since all points that lie in the same half plane as P are closer to P than they are to O, this is true in particular of that vertex of $\triangle ABC$ which is in the same half plane as P.

1907/3. Let

$$\frac{r}{s} = 0.k_1k_2k_3\cdots$$

be the decimal expansion of a rational number.† Prove that at least two of the numbers

$$\sigma_1 = 10\left(\frac{r}{s}\right) - k_1, \qquad \sigma_2 = 10^2\left(\frac{r}{s}\right) - (10k_1 + k_2),$$

$$\sigma_3 = 10^3\left(\frac{r}{s}\right) - (10^2k_1 + 10k_2 + k_3), \quad \cdots$$

are equal.

Solution. The number

$$0.k_1k_2\cdots k_m = \frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_m}{10^m}$$

is not greater than r/s, but if $1/10^m$ is added to it the resulting number is greater than r/s:

$$0 \leq \frac{r}{s} - \left(\frac{k_1}{10} + \frac{k_2}{10^2} + \cdots + \frac{k_m}{10^m}\right) < \frac{1}{10^m}.$$

For example, if $r/s = .235796\cdots$, we have

$$0 \leq .235 \leq .235796\cdots < .236$$

or

$$0 \leq .000796\cdots \leq .001.$$

† If this is a terminating decimal, all k_i from a certain one on are 0.

If we multiply these inequalities by 10^m, we obtain

$$0 \leq \frac{10^m r - s(10^{m-1}k_1 + 10^{m-2}k_2 + \cdots + k_m)}{s} = \sigma_m < 1.$$

Therefore the numerator of σ_m is an integer less than s, i.e., it is one of the numbers

$$0, 1, 2, \cdots, s - 1.$$

Since this is true for any m, the numbers $\sigma_1, \sigma_2, \cdots$ fall into s possible classes according to the value of the numerator. But then, according to the pigeonhole principle (cf. 1906/3, Note) even the first $s + 1$ numbers $\sigma_1, \sigma_2, \cdots, \sigma_{s+1}$ are not all distinct.

1908 Competition

1908/1. Given two odd integers a and b; prove that $a^3 - b^3$ is divisible by 2^n if and only if $a - b$ is divisible by 2^n.

Solution. a) If we set $A = a^2 + ab + b^2$ and $B = a - b$, then $a^3 - b^3 = AB$. If B is divisible by 2^n, then the product AB is divisible by 2^n.

b) To prove the "only if" part we observe that since a and b are odd, $A = a^2 + ab + b^2$ is a sum of three odd numbers and is therefore also odd. Hence, A is relatively prime to 2^n. Thus 2^n divides AB only if it divides B (cf. 1901/3, Note 3).

1908/2. Let n be an integer greater than 2. Prove that the nth power of the length of the hypotenuse of a right triangle is greater than the sum of the nth powers of the lengths of the legs.

Solution. The hypotenuse, c, is longer than either of the legs a, b. From $c > a$, $c > b$, it follows that $c^k > a^k$, $c^k > b^k$ for any positive exponent k. Using the Pythagorean theorem, we may write

$$c^n = c^2 \cdot c^{n-2} = (a^2 + b^2)c^{n-2} = a^2c^{n-2} + b^2c^{n-2}.$$

If we now replace c^{n-2} by the smaller quantities a^{n-2} and b^{n-2}, we obtain

$$c^n = a^2c^{n-2} + b^2c^{n-2} > a^2a^{n-2} + b^2b^{n-2} = a^n + b^n,$$

so that

$$c^n > a^n + b^n.$$

1908/3. A regular polygon of 10 sides (a regular decagon) may be inscribed in a circle in the following two distinct ways: Divide the circumference into 10 equal arcs and (1) join each division point to the next by straight line segments, (2) join each division point to the next but two by straight line segments. (See Figures 48 and 49.) Prove that the difference in the side lengths of these two decagons is equal to the radius of their circumscribed circle.

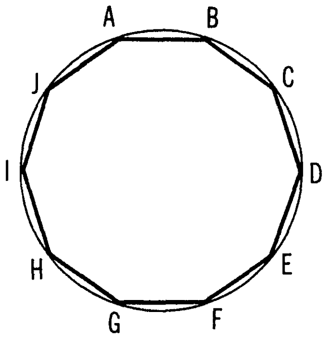

Figure 48

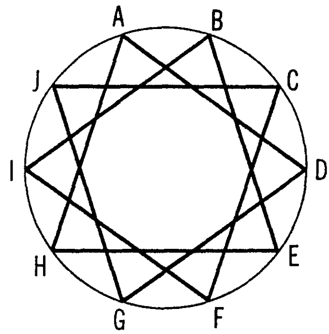

Figure 49

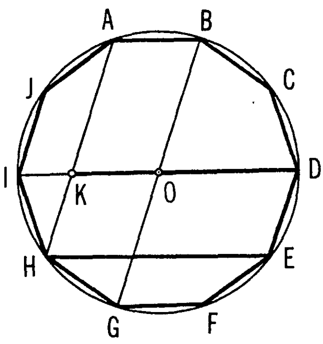

Figure 50

Solution. In Figure 50, the diameter IOD is parallel to the side AB of the simple decagon and to the side HE of the self-intersecting decagon. Similarly, the diameter BOG is parallel to DE and to AH. We therefore have parallelograms $ABOK$ and $DEHK$ with

$$AB = KO \quad \text{and} \quad HE = KD,$$

so that

$$HE - AB = KD - KO = OD$$

which is the radius of the circle.

1909 Competition

1909/1. Consider any three consecutive natural numbers. Prove that the cube of the largest cannot be the sum of the cubes of the other two.

Solution. Let $n - 1$, n, $n + 1$ be the consecutive numbers. If the equation

$$(n + 1)^3 = n^3 + (n - 1)^3,$$

that is,

$$n^3 + 3n^2 + 3n + 1 = n^3 + n^3 - 3n^2 + 3n - 1,$$

were satisfied, then

$$2 = n^2(n - 6).$$

But the right side is positive only if $n > 6$, and in that case

$$n^2(n - 6) > 36$$

and therefore cannot be 2.

Note. Fermat's conjecture. The theorem just proved is the simplest instance of the following statement made by Fermat:

If n is an integer greater than 2, the equation

$$x^n + y^n = z^n$$

has no positive integer solution (x, y, z). The statement is known to be true for $n < 2003$ and, as of 1961, for all prime exponents less than 4002.† In spite of the efforts of many distinguished mathematicians, no proof has been found for Fermat's conjecture to this day.

Certain special cases are easily proved; the following, for example, includes the problem given above.

If n is an odd integer greater than 1, then three consecutive terms x, y, z of an arithmetic progression never satisfy

$$x^n + y^n = z^n.$$

Let $x = y - d$, $z = y + d$, where y and d are positive integers, $y > d$. If we divide

$$(y - d)^n + y^n = (y + d)^n \tag{1}$$

by d^n and set $t = y/d$, we obtain the equation

$$(t - 1)^n + t^n = (t + 1)^n,$$

† See "Proof of Fermat's Last Theorem for All Prime Exponents Less than 4002," by H. S. Vandiver, J. L. Selfridge and C. A. Nicol, *Proceedings of the National Academy of Science*, Vol. 41, pp. 970–973 (1955).

or

$$t^n - \binom{n}{1} t^{n-1} + \binom{n}{2} t^{n-2} - \cdots - 1 + t^n$$
$$= t^n + \binom{n}{1} t^{n-1} + \binom{n}{2} t^{n-2} + \cdots + 1,$$

which reduces to the equation

$$t^n - 2\binom{n}{1} t^{n-1} - 2\binom{n}{3} t^{n-3} - \cdots - 2 = 0 \tag{2}$$

for the rational number $t = y/d$. Since the leading coefficient is 1 and all other coefficients are integers, every rational root of this equation is an integer (see 1907/1, Note). However (2) is not satisfied by any odd integer [since all terms except the first in (2) would be divisible by 2], and also not by any even integer [since for n greater than 1, all but the last term of (2) would be divisible by 4, cf. also 1907/1]. It follows that (2) is satisfied by no rational number and hence that (1) is satisfied by no integer y.

1909/2. Show that the radian measure of an acute angle is less than the arithmetic mean of its sine and its tangent.

First Solution. We must prove the inequality

$$\varphi < \tfrac{1}{2}[\sin\varphi + \tan\varphi].$$

In Figure 51, let $\sphericalangle AOB$ be acute and take $OA = OB = 1$, so that the arc AB of the unit circle with center O has length $\varphi < \pi/2$. Let C be the intersection of the tangents to the circle at A and B, and let D be the intersection of lines AC and OB extended.

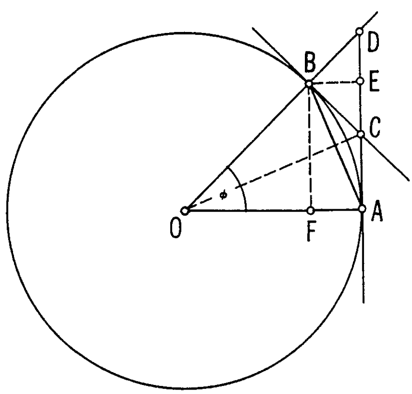

Figure 51

The sector OAB and the triangles OAB, OAD have areas

$$\frac{\varphi}{2}, \qquad \frac{\sin\varphi}{2}, \qquad \frac{\tan\varphi}{2}$$

respectively; if we can show, for these areas, that

$$\text{(1)} \qquad \text{sector } OAB < \tfrac{1}{2}[\triangle OAB + \triangle OAD],$$

we shall have solved the problem.

We shall prove that

$$\text{(2)} \qquad \text{quadrilateral } OACB < \tfrac{1}{2}[\triangle OAB + \triangle OAD],$$

and since quadrilateral $OACB$ contains sector OAB, the inequality (1) will follow.

Inequality (2) may be written (after multiplication by 2 and rearranging) as

$$\text{quadrilateral } OACB - \triangle OAB < \triangle OAD - \text{quadrilateral } OACB$$

which, see Figure 51, is equivalent to

$$\text{(3)} \qquad \triangle ACB < \triangle CDB.$$

These triangles have a common altitude BE, where E denotes the foot of the perpendicular from B to AD. Consequently, (3) is equivalent to

$$AC < CD.$$

But this last inequality is true because the hypotenuse CD of the right triangle CDB is longer than its leg BC which is equal to AC.

Second Solution. We shall prove somewhat more than stated in the problem, namely: that *the radian measure of an acute angle is less than the harmonic mean of its sine and its tangent.* This is really a stronger statement since the arithmetic mean

$$\tfrac{1}{2}(a + b)$$

of two unequal positive numbers a and b is greater than their harmonic mean

$$\frac{2}{\frac{1}{a} + \frac{1}{b}},$$

cf. Note below.

The harmonic mean of $\sin\varphi$ and $\tan\varphi$ is

$$\frac{2}{\dfrac{1}{\sin\varphi}+\dfrac{1}{\tan\varphi}}=\frac{2\sin\varphi}{1+\cos\varphi}=\frac{4\sin(\varphi/2)\cos(\varphi/2)}{2\cos^2(\varphi/2)}=2\tan\frac{\varphi}{2};$$

so we must prove that

$$\varphi < 2\tan\frac{\varphi}{2}.$$

But we already noted in the First Solution that the sector OAB is contained in quadrilateral $OACB$, i.e., that

$$\text{(4)}\qquad \text{sector } OAB < \text{quadrilateral } OACB = 2\triangle OAC.$$

Since the area of the sector OAB is $\frac{1}{2}\varphi$ and since the area of $\triangle OAC$ is $\frac{1}{2}\tan(\varphi/2)$ we have

$$\frac{\varphi}{2} < 2\cdot\frac{1}{2}\tan\frac{\varphi}{2}$$

from which

$$\varphi < 2\tan\frac{\varphi}{2}$$

follows.†

Note. The inequality between the arithmetic and harmonic means of two positive numbers. To prove that

$$\frac{a+b}{2} > \frac{2}{\dfrac{1}{a}+\dfrac{1}{b}},\qquad \text{for } a>0,\quad b>0,\quad \text{and}\quad a\neq b,$$

† From the fact that $\triangle OAB$ (see Figure 51) is contained in sector OAB which, in turn, is contained in $\triangle OAD$ it follows that

$$2\triangle OBF = \sin\varphi < 2 \text{ sector } OAB = \varphi < 2\triangle OAD = \tan\varphi,$$

i.e. $\sin\varphi < \varphi < \tan\varphi$, and our result tells how an acute angle may be estimated in terms of the arithmetic (or harmonic) means of its upper and lower bounds.

we show that the difference

$$\frac{a+b}{2}-\frac{2}{\dfrac{1}{a}+\dfrac{1}{b}}=\frac{a+b}{2}-\frac{2ab}{a+b}=\frac{(a+b)^2-4ab}{2(a+b)}=\frac{(a-b)^2}{2(a+b)}$$

is always positive. Clearly, $(a-b)^2$ is positive and so is the denominator. Observe that if the case $a=b$ is admitted, the above difference is zero and the arithmetic and harmonic means of a and b are equal; this is, of course, to be expected since a mean of two numbers always lies between them.

1909/3. Let A_1, B_1, C_1 be the feet of the altitudes of $\triangle ABC$ drawn from the vertices $A, B, C,$ respectively, and let M be the orthocenter (point of intersection of altitudes) of $\triangle ABC$. Assume that the orthic triangle† $A_1B_1C_1$ exists. Prove that each of the points M, A, B, and C is the center of a circle tangent to all three sides (extended if necessary) of $\triangle A_1B_1C_1$. What is the difference in the behavior of acute and obtuse triangles ABC?

Solution. If ABC is a right triangle, then the feet of two of its altitudes coincide at a vertex, so there is no orthic triangle. For acute and obtuse triangles the problem is solved completely in 1896/3, Solution, a), b) and Note.

1910 Competition

1910/1. If a, b, c are real numbers such that

$$a^2+b^2+c^2=1,$$

prove the inequalities

$$-\tfrac{1}{2}\le ab+bc+ca\le 1.$$

Solution. Since $a^2+b^2+c^2=1$, the inequalities to be proved may be written in the form

$$-\tfrac{1}{2}(a^2+b^2+c^2)\le ab+bc+ca\le a^2+b^2+c^2$$

† The triangle whose vertices are the feet of the altitudes of the original triangle. (See also footnote to 1896/3.)

or

$$-(a^2 + b^2 + c^2) \leq 2(ab + bc + ca) \leq 2(a^2 + b^2 + c^2).$$

These inequalities are indeed true since

$$2(ab + bc + ca) + (a^2 + b^2 + c^2) = (a + b + c)^2 \geq 0,$$

and

$$2(a^2 + b^2 + c^2) - 2(ab + bc + ca)$$
$$= (a - b)^2 + (a - c)^2 + (b - c)^2 \geq 0.$$

1910/2. Let a, b, c, d and u be integers such that each of the numbers

$$ac, \qquad bc + ad, \qquad bd$$

is a multiple of u. Show that bc and ad are multiples of u.

First Solution. Think of every positive integer as a product of powers of primes; thus

$$u = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

is uniquely represented and contains each prime p_i exactly to that power α_i for which u is divisible by $p_i^{\alpha_i}$ and not by $p_i^{\alpha_i+1}$ (cf. 1896/1, Note). In view of this and 1901/3, Note 1, the above problem is equivalent to the following:

If $u = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *divides each of the integers*

$$(1) \qquad ac, \qquad bc + ad, \qquad bd,$$

then the prime power products for bc *and* ad *contain* p_i *to powers not less than* α_i, $i = 1, 2, \cdots, k$.

Let p be any of the primes p_i occurring in the prime power product of u. Since ac, $bc + ad$, and bd are multiples of u and since u is a multiple of $p^r = p_i^{\alpha_i}$, we have

$$(2) \qquad ac = p^r A, \qquad bc + ad = p^r B, \qquad bd = p^r C,$$

for some integers A, B, C. Therefore

$$(bc)(ad) = (ac)(bd) = p^{2r} AC.$$

This relation can hold only if p occurs to the power $\alpha \geq r$ either in bc or in ad. But if this is so, then the second equation in (2) implies

that p occurs to at least the power r in bc *and* in ad. Therefore bc and ad are multiples of p^r.

This reasoning applies to each prime that occurs in the factorization of u. It follows that bc and ad are multiples of u.

Second Solution. The identity

$$(bc - ad)^2 = (bc + ad)^2 - 4abcd$$

implies that

$$\left(\frac{bc - ad}{u}\right)^2 = \left(\frac{bc + ad}{u}\right)^2 - 4\frac{ac}{u} \cdot \frac{bd}{u}, \tag{3}$$

where the right side is an integer. The left side is an integer only if

$$\frac{bc - ad}{u}$$

is an integer (cf. 1907/1, Solution and Note). Moreover, since by (3), the difference of the squares of the numbers

$$s = \frac{bc + ad}{u}, \qquad t = \frac{bc - ad}{u}$$

is even, s and t are either both even or both odd. Hence

$$\frac{s + t}{2} = \frac{bc}{u}, \qquad \frac{s - t}{2} = \frac{ad}{u}$$

are integers; that is, bc and ad are multiples of u.

1910/3. The lengths of sides CB and CA of $\triangle ABC$ are a and b, and the angle between them is $\gamma = 120°$. Express the length of the bisector of γ in terms of a and b.

First Solution. In the arbitrary triangle ABC, let v denote the length of the bisector CD of $\sphericalangle ACB = \gamma$, see Figure 52. Since the area of $\triangle ABC$ is the sum of the areas of triangles ACD and CDB, we have

$$ab \sin \gamma = av \sin \frac{\gamma}{2} + bv \sin \frac{\gamma}{2} = v(a + b) \sin \frac{\gamma}{2},$$

and since

$$\sin\gamma = 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2},$$

it follows that

$$2ab\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} = v(a+b)\sin\frac{\gamma}{2};$$

dividing this equation by $2abv\sin(\gamma/2)$, we obtain

$$\frac{1}{v}\cos\frac{\gamma}{2} = \frac{1}{2}\frac{a+b}{ab} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right).$$

If $\gamma = 120°$, $\cos\gamma/2 = \cos 60° = \frac{1}{2}$ so that

$$\frac{1}{v} = \frac{1}{a}+\frac{1}{b} \qquad \text{or} \qquad v = \frac{ab}{a+b}.$$

In words: v is half the harmonic mean† of a and b.

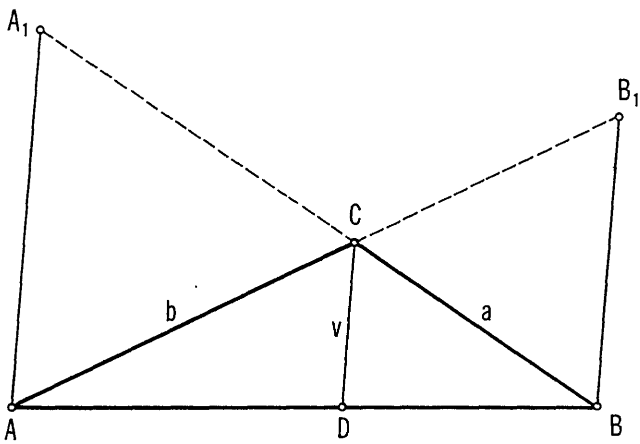

Figure 52

Second Solution. Let D be any point on side AB of $\triangle ABC$. Through A and B, draw lines parallel to CD and let A_1, B_1 be their

† If a and b are two positive numbers, their *harmonic mean* c is defined by

$$\frac{1}{a}+\frac{1}{b}=\frac{2}{c},$$

i.e. it is the number whose reciprocal is the average of the reciprocals of a and b; see also pp. 33, 34.

intersections with the extensions of lines BC and AC, respectively. Then, by 1905/3,

(1) $$\frac{1}{CD} = \frac{1}{AA_1} + \frac{1}{BB_1}.$$

If $\sphericalangle ACB = 120°$ and if CD bisects it (Figure 52), then

$$\sphericalangle B_1BC = \sphericalangle BCD = 60°,$$

$$\sphericalangle BB_1C = \sphericalangle DCA = 60°,$$

so ΔBCB_1 is equilateral. Similarly we can show that ΔACA_1 is equilateral. Therefore

$$BB_1 = BC \qquad \text{and} \qquad AA_1 = AC$$

so that the equality (1) becomes

(2) $$\frac{1}{CD} = \frac{1}{AC} + \frac{1}{BC}$$

which is just

(3) $$\frac{1}{v} = \frac{1}{a} + \frac{1}{b}.$$

Note. On nomography: a) *The optical lens.* Equation (3) has the same form as

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f}$$

which describes the relationship in optics between the distance d of an object from a lens, the distance d' of the image from the lens, and the focal length f of the lens. The solution of the above problem, therefore, affords the following simple construction of a diagram with the help of which one of the quantities d, d', f can be found if the other two are given: From a point O, draw lines that form a 120° angle at O, draw the bisector of that angle, and mark off equal distances along these three lines using O as the initial point (see Figure 53). The quantities d and d' will be represented by points on the sides of the 120° angle, and f by a point on the bisector.

If d and d' are given we draw the line through the points representing the given values for d and d'; the intersection of this line with the bisector determines the corresponding value of f.

In practice, the line is drawn on transparent paper and can be laid over the diagram in the required position.

A *nomogram* (*nomos* = law, *gramma* = drawing) is a graph by means of which the magnitude of a quantity is easily read off when the magnitudes of other quantities are known. *Nomography* deals with the preparation of nomograms. The most convenient nomogram for three variables consists of graduated curves, called scales, such that three related values always lie on a straight line. A nomogram of this type is called an "alignment chart"; Figure 53 is of this kind.

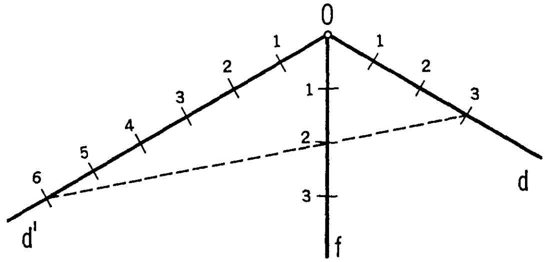

Figure 53

b) *The nomogram of a quadratic equation.* For each pair of values p, q, the nomogram of Figure 54 supplies the real solutions of the quadratic equation

$$z^2 + pz + q = 0. \tag{1}$$

The construction is based on the known fact of analytic geometry† that the straight line

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{2}$$

intersects the coordinate axes at a and b, respectively. In other words, equation (2) says that the points

$$(a, 0), \qquad (0, b), \qquad (x, y)$$

are collinear.

Now we divide equation (1) by z^2 and write it as follows:

$$1 = -p\frac{1}{z} - q\frac{1}{z^2} = -\frac{p}{\alpha}\cdot\frac{\alpha}{z} - \frac{q}{\beta}\cdot\frac{\beta}{z^2},$$

where α, β are *arbitrarily chosen constants.* The last equation has the same

† Descartes is traditionally credited with the founding of analytic geometry. His great contemporary, Fermat, also discovered and used this method of studying geometry, but his work on the subject did not appear until after his death in 1665 at which time Descartes' work had already been published.

R. Descartes (latinized: *Cartesius*), French philosopher, mathematician, and physicist, was born in 1596 and died in Stockholm in 1650.

form as (2), with

$$a, \quad b, \quad x, \quad y \cdot$$

replaced by

$$\frac{\alpha}{p}, \qquad \frac{\beta}{q}, \qquad -\frac{\alpha}{z}, \qquad -\frac{\beta}{z^2};$$

in other words, *equation* (2) *expresses the condition that the points*

$$\left(\frac{\alpha}{p}, \quad 0\right), \qquad \left(0, \quad \frac{\beta}{q}\right), \qquad \left(-\frac{\alpha}{z}, \quad -\frac{\beta}{z^2}\right)$$

be collinear.

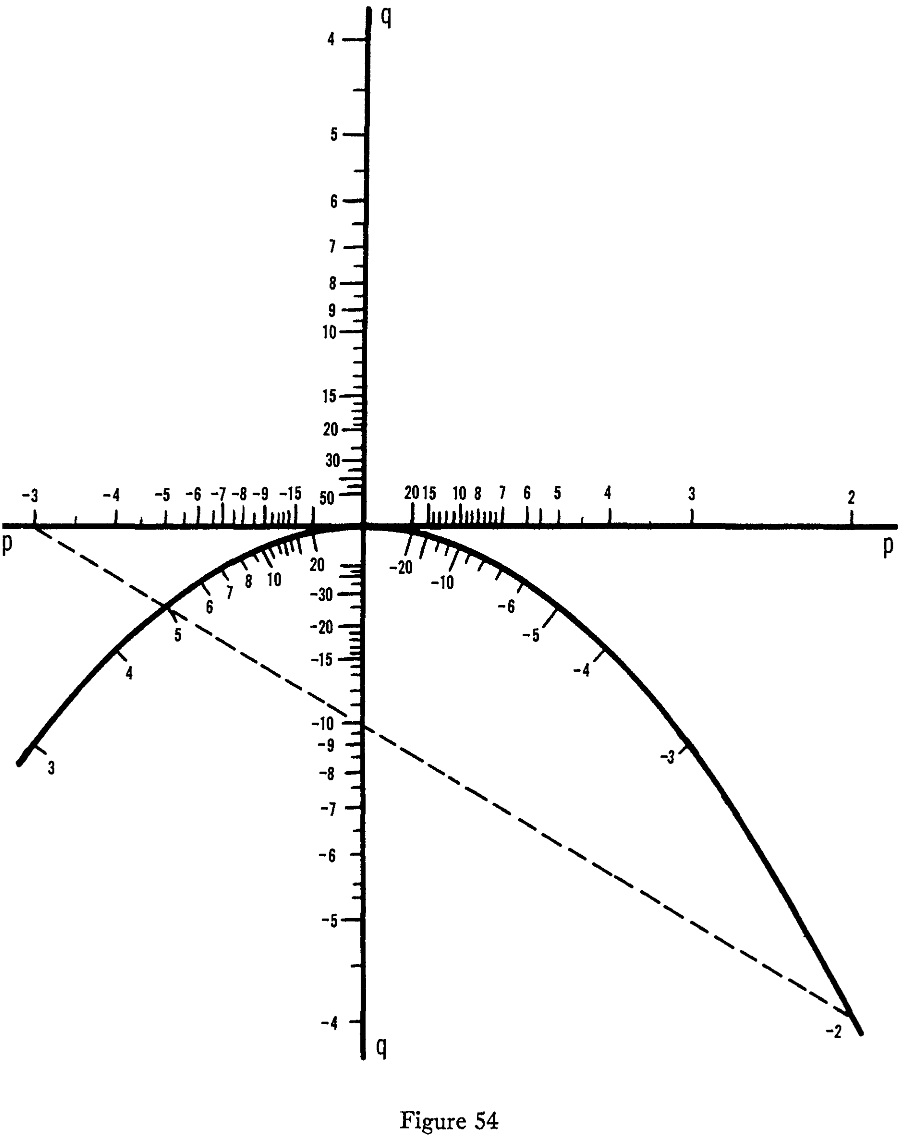

Figure 54

We now use the following three scales for our nomogram: The x-axis will be the scale of p marked so that for a given value of p, the point

$$\left(\frac{\alpha}{p},\ 0\right)$$

is used to represent p; the y-axis will be the scale of q with the point

$$\left(0,\ \frac{\beta}{q}\right)$$

representing the value q; the point

$$\left(-\frac{\alpha}{z},\ -\frac{\beta}{z^2}\right)$$

in the plane will represent z. The locus of these points is obtained by eliminating z from

$$x = -\frac{\alpha}{z}, \qquad y = -\frac{\beta}{z^2};$$

it is the parabola

$$y = -\frac{\beta}{\alpha^2}x^2.$$

Fig. 54 was obtained in this way (with $\alpha = 12$, $\beta = 24$, unit length = 1 cm.).

If, for example, we want to find the roots of

$$z^2 - 3z - 10 = 0,$$

we draw the line through $p = -3$, $q = -10$ and find its intersection with the scale of z. We obtain

$$z_1 = 5, \qquad z_2 = -2.$$

If the quadratic equation to be solved has the coefficients $p = -6.4$, $q = 4.9$ (perhaps approximately), then this method gives $z_1 = 5.5$ (approximately). Since $z_1 + z_2 = 6.4$, we can deduce that $z_2 = 0.9$ (approximately) although this number falls outside the drawn portion of the nomogram.†

It is noteworthy that this nomogram can perform multiplications. For example, the product of the numbers

$$z_1 = -2, \qquad z_2 = 5$$

† In practice, the part actually to be drawn and the accuracy with which the points of the scales are to be drawn depend on the relevant range of the variables and the required accuracy for the solution.

can be obtained by drawing the line through $z_1 = -2$ and $z_2 = 5$ and finding its intersection with the scale of q. The result is $q = -10$.

The founder of nomography as a separate discipline is d'Ocagne.†

1911 Competition

1911/1. Show that, if the real numbers a, b, c, A, B, C satisfy

$$aC - 2bB + cA = 0 \quad \text{and} \quad ac - b^2 > 0.$$

then

$$AC - B^2 \leq 0.$$

Solution. We must show that the relations

$$aC + cA = 2bB, \tag{1}$$

$$ac > b^2, \tag{2}$$

$$AC > B^2 \tag{3}$$

are *incompatible.*

From (2) and (3), we obtain

$$acAC > b^2B^2,$$

and by squaring (1), we obtain

$$(aC + cA)^2 = 4b^2B^2$$

so that

$$4acAC > 4b^2B^2 = (aC + cA)^2 = a^2C^2 + 2acAC + c^2A^2$$

or

$$0 > (aC - cA)^2.$$

But this is impossible since the square of a real number is never negative.

† M. d'Ocagne, 1862–1938, was professor of geometry at the *École Polytechnique* in Paris. His pioneering work on alignment charts appeared in 1884 in the yearbook of the *École des ponts et chaussées*, where he was a student at the time. His subsequent work published in numerous articles and books made alignment charts a widely used tool, specially in technical work. Although a relation between three variables cannot always be given by an alignment chart, such a representation does exist in many important practical applications.

1911/2. Let Q be any point on a circle and let $P_1P_2P_3\cdots P_8$ be a regular inscribed octagon. Prove that the sum of the fourth powers of the distances from Q to the diameters P_1P_5, P_2P_6, P_3P_7, P_4P_8 is independent of the position of Q.

First Solution. Suppose Q is on the arc P_2P_3 of the circle; see Figure 55. Let A, B, C, and D denote the feet of the perpendiculars from Q to P_1P_5, P_2P_6, P_3P_7 and P_4P_8, and let O be the center of the octagon. Since angles QAO, QBO, QCO and QDO are right angles, the quadrilateral $ABCD$ can be inscribed in a circle k with diameter OQ, and since $\measuredangle AOB = \measuredangle BOC = \measuredangle COD = 45°$, $ABCD$ is a square. The diameter of the circle k is equal to the radius of the given circle, so the size of the square $ABCD$ does not depend on Q.

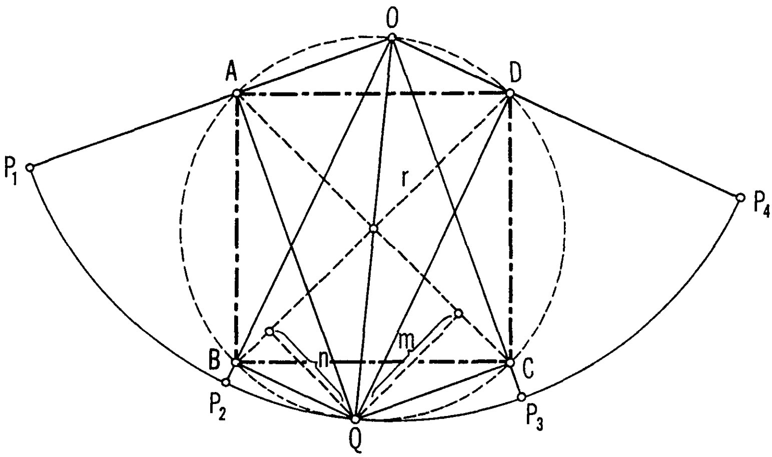

Figure 55

Thus, it now suffices to prove: *If Q is a point on a circle circumscribed about a square, the sum of the fourth powers of the distances from Q to the vertices is independent of the position of Q.*

Let S denote the sum under consideration:

$$\begin{aligned} S &= (QA)^4 + (QB)^4 + (QC)^4 + (QD)^4 \\ &= [(QA)^2 + (QC)^2]^2 + [(QB)^2 + (QD)^2]^2 \\ &\qquad - 2[(QA \cdot QC)^2 + (QB \cdot QD)^2]. \end{aligned}$$

If r is the length of the diagonal of the square, then by Pythagoras' theorem

$$S = (r^2)^2 + (r^2)^2 - 2[(QA \cdot QC)^2 + (QB \cdot QD)^2].$$

The brackets contain the sum of the squares of twice the areas of the

right triangles AQC and BQD. Let m and n be the lengths of the altitudes from Q of these triangles and observe that they form two sides of a rectangle (the other two sides of which lie along AC and BD) whose diagonal is the radius $r/2$ of the circle k, so that $m^2 + n^2 = (r/2)^2$. We now express the areas as follows:

$$(QA \cdot QC)^2 + (QB \cdot QD)^2 = (AC \cdot m)^2 + (BD \cdot n)^2$$
$$= r^2(m^2 + n^2) = r^2\left(\frac{r}{2}\right)^2,$$

and hence

$$S = 2r^4 - 2\frac{r^4}{4} = \frac{3}{2}r^4,$$

which depends only on the radius r of the given circle.

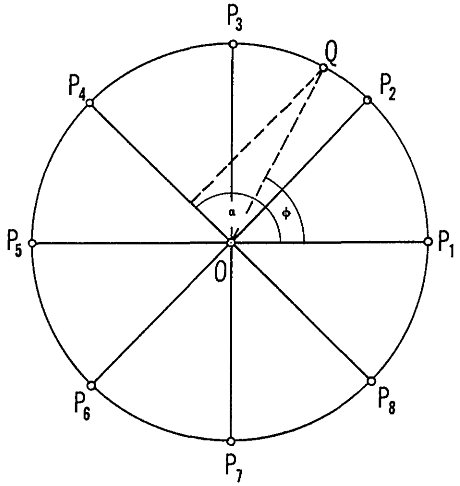

Figure 56

Second Solution. Without loss of generality we assume that the radius of the given circle with center O is 1. Suppose OQ, OP_1, OP_2, OP_3, OP_4 make angles φ, α_1, α_2, α_3, α_4 with the ray OP_1; see Figure 56. Then

$$\alpha_1 = 0, \quad \alpha_2 = 45°, \quad \alpha_3 = 90°, \quad \alpha_4 = 135°.$$

The distance d between Q and any line OP that makes an angle α with OP_1 is given by

$$d = |\sin(\varphi - \alpha)|;$$

therefore, the sum in question is

$$f(\varphi) = \sin^4(\varphi - \alpha_1) + \sin^4(\varphi - \alpha_2) + \sin^4(\varphi - \alpha_3) + \sin^4(\varphi - \alpha_4).$$

Let us express these fourth powers as sums (cf. 1897/1, Note):

$$\sin^4 x = (\sin^2 x)^2 = [\tfrac{1}{2}(1 - \cos 2x)]^2 = \tfrac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$
$$= \tfrac{1}{4}[1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)]$$

so that

$$\sin^4 x = \tfrac{3}{8} - \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x. \tag{1}$$

We calculate expression (1) using the values

$$\varphi - 0, \quad \varphi - 45°, \quad \varphi - 90°, \quad \varphi - 135°,$$

one after the other, for x. In order to obtain the sum $f(\varphi)$, we then add these four expressions, make use of the relation

$$\cos(z + 180°) = -\cos z$$

to cancel pairs of cosine terms, and finally obtain

$$f(\varphi) = \tfrac{3}{2}.$$

Note. A theorem on trigonometric polynomials. The function

$$p(\varphi) = a_0 + a_1 \cos\varphi + b_1 \sin\varphi + \cdots + a_k \cos k\varphi + b_k \sin k\varphi,$$

of the angle φ [of which the expression (1) is a special case] is a *trigonometric polynomial* of degree k. The following general theorem holds, but will not be proved here:

If $p(\varphi)$ is a trigonometric polynomial of degree k with constant term a_0, and if $\theta = 2\pi/n$ for $n > k$, then

$$p(\varphi + \theta) + p(\varphi + 2\theta) + \cdots + p(\varphi + n\theta) = na_0.$$

Our solution made use of the special case $k = 4$, $n = 8$ of this theorem.

1911/3. Prove that $3^n + 1$ is not divisible by 2^n for any integer $n > 1$.

Solution. We shall prove a more general theorem: *If n is even, then 2^1 is the highest power of 2 that divides $3^n + 1$; if n is odd, then 2^2 is the highest power of 2 that divides $3^n + 1$.*

We make use of the observation that *the square of an odd number is* 1

plus a multiple of 8; for, if the odd number is

$$a = 2k + 1,$$

then

$$a^2 = 4k(k + 1) + 1,$$

and since either k or $k + 1$ is even, the first summand is a multiple of 8. We now apply this together with the fact that any power of an odd number is odd to the problem at hand.

If n is even, that is $n = 2m$, then

$$3^n = 3^{2m} = (3^m)^2 = 8b + 1 \qquad (b \text{ an integer}),$$

hence

$$3^n + 1 = 8b + 2 = 2(4b + 1).$$

If n is odd, that is $n = 2m + 1$, then

$$3^n + 1 = 3^{2m+1} + 1 = 3 \cdot 3^{2m} + 1 = 3(8b + 1) + 1 = 4(6b + 1).$$

In both cases, the sum in parentheses is an odd number, so the theorem is proved.

1912 Competition

1912/1. How many positive integers of n digits exist such that each digit is 1, 2, or 3? How many of these contain all three of the digits 1, 2, and 3 at least once?

Solution. The answer to the first question is the number of permutations of three things (where repetitions are allowed), n at a time, namely 3^n.

To find the number of permutations that actually contain all three of these digits, we must subtract from 3^n

a) the number of positive integers whose digits take on only two of the three possible values, and

b) the three numbers all of whose digits are 1, 2, or 3, respectively; i.e. whose digits take on only one of the three possible values.

The number of cases in a) is $3(2^n - 2)$ (cf. Solution of 1895/1). Hence

$$3^n - 3(2^n - 2) - 3 = 3^n - 3 \cdot 2^n + 3$$

of the integers contain 1, 2, and 3 at least once.

Note. (i) *Classification of a set of objects according to several properties.* In order to recognize the essence of this problem, we shall consider the following more general and more abstract question.

Let a, b, c designate different properties of a set of N objects; let N_a be the number of those objects among the N that have *at least* property a (whether or not they have properties b or c); N_{ab} the number of those having at least properties a *and* b; N_{abc} the number of those having all three properties. Define similarly the symbols N_b, N_c, N_{bc}, N_{ac}. Question: If the values of N, N_a, N_b, N_c, N_{ab}, N_{ac}, N_{bc}, N_{abc} are given, how can we find out how many objects have none of the properties a, b, c?

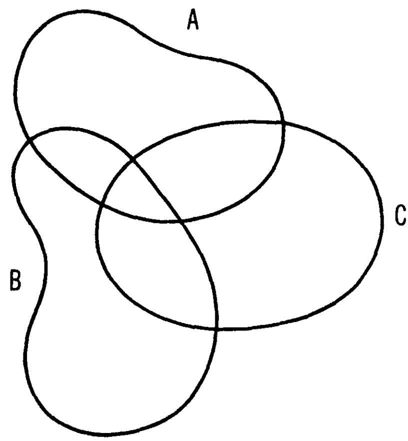

Figure 57

This situation has the following schematic formulation (Figure 57):

Let the objects be N points in the plane, and let a be the property of "being inside or on the closed curve A". Define b and B, c and C similarly. Then the problem is to calculate the number of points outside each closed curve if the number of points

inside or on A,	inside or on A and B,
inside or on B,	inside or on A and C,
inside or on C,	inside or on B and C,

inside or on A, B, and C

is known.

We divide our task into easy steps.

α) How many objects do not have property a? In the geometric analogue: how many points are outside A? Clearly, this number is

$$N - N_a. \tag{1}$$

β) Of the N_b objects enjoying property b, how many do not have property a? This question reduces to the question asked in α) when N is re-

placed by N_b and N_a by N_{ab}; hence the answer is

$$N_b - N_{ab}. \tag{2}$$

(What is the geometrical analogue?)

γ) How many objects have neither property a nor property b? To find this number, we must remove from the $N - N_a$ objects calculated in α) the $N_b - N_{ab}$ objects we calculated in β); there are

$$N - N_a - (N_b - N_{ab}) = N - N_a - N_b + N_{ab} \tag{3}$$

objects left. (What is the geometrical analogue?)

δ) Of the N_c objects enjoying property c, how many fail to have properties a, b ? This reduces to the question asked in γ) when N is replaced by N_c; therefore the answer is

$$N_c - N_{ac} - N_{bc} + N_{abc}. \tag{4}$$

(What is the geometrical analogue?)

ϵ) The number of objects having none of the properties a, b, c is

$$\begin{aligned} &N - N_a - N_b + N_{ab} - (N_c - N_{ac} - N_{bc} + N_{abc}) \\ &\qquad = N - N_a - N_b - N_c + N_{ab} + N_{ac} + N_{bc} - N_{abc}. \end{aligned} \tag{5}$$

(ii) *Application to the problem.* The given objects are n-digit positive integers containing only the digits 1, 2, and 3. The property

a signifies the absence of 1's;

b signifies the absence of 2's;

c signifies the absence of 3's.

Then

$$N = 3^n, \quad N_a = N_b = N_c = 2^n, \quad N_{ab} = N_{ac} = N_{bc} = 1, \quad N_{abc} = 0,$$

so that (5) has the value

$$3^n - 3 \cdot 2^n + 3$$

found before.

(iii) *A generalization.* Formula (5) can be generalized to apply to any number of properties, and the following theorem then becomes clear:

There are exactly

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots + (-1)^{k-1}\binom{k}{k-1}$$

n-digit numbers made up of and actually containing the digits 1, 2, 3, $\cdots$, k.

If $k > 9$, these arrays of symbols can no longer be interpreted as integers in decimal notation; but they can be so interpreted in a number system to a larger base.

For $n = k$ we obtain an interesting identity; in this case the n-digit numbers made up of and actually containing the digits $1, 2, \cdots, k(= n)$ just represent all the permutations of the integers $1, 2, \cdots, n$, so that

$$n^n - \binom{n}{1}(n-1)^n + \binom{n}{2}(n-2)^n - \cdots + (-1)^{n-1}\binom{n}{n-1} = n!.$$

1912/2. Prove that for every positive integer n, the number

$$A_n = 5^n + 2 \cdot 3^{n-1} + 1$$

is a multiple of 8.

First Solution. $A_1 = 5^1 + 2 \cdot 3^0 + 1 = 8$. This shows that the assertion is true for $n = 1$. We shall show that, if it is true for A_k, then it is true for A_{k+1} (complete induction).

Since

$$A_k = 5^k + 2 \cdot 3^{k-1} + 1,$$

we find that

$$A_{k+1} = 5^{k+1} + 2 \cdot 3^k + 1$$
$$= 5 \cdot 5^k + 6 \cdot 3^{k-1} + 1.$$

Therefore

$$A_{k+1} - A_k = (5-1)5^k + (6-2)3^{k-1} = 4(5^k + 3^{k-1}).$$

Since 5 and 3 are odd, 5^k and 3^{k-1} are also odd and their sum is even. Hence $A_{k+1} - A_k$ is a multiple of 8; and since A_k is a multiple of 8 (by the induction hypothesis), it follows that A_{k+1} is a multiple of 8.

Second Solution. A_n may be written in the forms

$$A_n = (5^n + 3^n) - (3^{n-1} - 1) = 5(5^{n-1} + 3^{n-1}) - (3^n - 1).$$

We use the first form if n is odd and the second if n is even. In both cases the first summand is divisible by $5 + 3 = 8$ and the second by $3^2 - 1 = 8$ because $a + b$ divides $a^k + b^k$ if k is odd, and $c^2 - 1$ divides $c^{2k} - 1$. Consequently, A_n is always divisible by 8.

1912/3. Prove that the diagonals of a quadrilateral are perpendicular if and only if the sum of the squares of one pair of opposite sides equals that of the other.

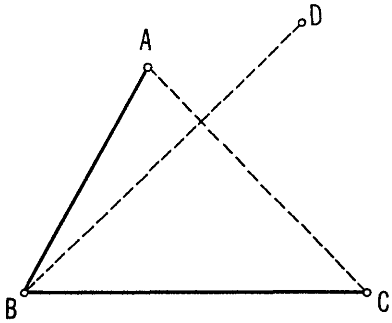

Figure 58

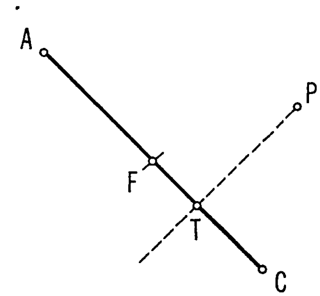

Figure 59

First Solution. Our task is to show that the diagonals BD and AC of quadrilateral $ABCD$ are perpendicular if and only if

$$(AB)^2 + (CD)^2 = (AD)^2 + (CB)^2$$

or, equivalently, if and only if

$$(AB)^2 - (CB)^2 = (AD)^2 - (CD)^2. \tag{1}$$

Let us keep points A, B, C fixed (Figure 58) so that the left member of (1) is constant, and consider the set of all points D that satisfy (1). B is such a point [for, when D is B, (1) is obviously satisfied]. Thus the assertion to be proved is equivalent to the theorem:

Let A and C be two given points. The locus of points P in the plane for which $(PA)^2 - (PC)^2$ is a fixed constant q is a line perpendicular to AC.

We shall prove this by showing that the value $q = (PA)^2 - (PC)^2$ depends only on the foot T of the perpendicular from P to AC and changes with the position of T along AC; see Figure 59. By Pythagoras' theorem,

$$\begin{aligned} q = (PA)^2 - (PC)^2 &= [(TA)^2 + (TP)^2] - [(TC)^2 + (TP)^2] \\ &= (TA)^2 - (TC)^2. \end{aligned}$$

This value of q cannot be the same for two different positions of T along AC. To see this, it is sufficient to consider only points T on one side of the midpoint F of AC (since for points T on opposite sides of F, q has different signs and for $T = F$, q is zero). For example, for T on the segment FC (see Figure 59),

$$\begin{aligned} q = (TA)^2 - (TC)^2 &= (AF + FT)^2 - (FC - FT)^2 \\ &= (AF + FT)^2 - (AF - FT)^2 \\ &= 4AF \cdot FT. \end{aligned}$$

Since AF is fixed, q clearly depends on the distance FT and this distance is different for different positions of T along FC.

The assertion of the problem is now easily proved with the aid of this theorem. For, the relation

$$(AB)^2 + (CD)^2 = (AD)^2 + (CB)^2,$$

and hence

$$(AB)^2 - (CB)^2 = (AD)^2 - (CD)^2$$

holds if and only if the points B and D lie on the same perpendicular to AC, that is, if and only if the diagonals of the quadrilateral $ABCD$ are perpendicular.

It follows from the assertion of the problem that, if a quadrilateral has pinned joints at the vertices, and if its diagonals are perpendicular in one position, then they are perpendicular in all positions of the quadrilateral.

Second Solution. a) We carry out the proof first for convex† quadrilaterals; we use the fact that, *in a triangle, the square of a side opposite an acute angle (opposite an obtuse angle) is less than (greater than) the sum of the squares of the remaining sides.* (This follows from the law of cosines, but can be deduced without trigonometry. See Note below.)

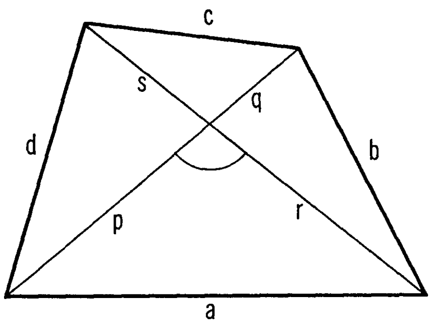

Figure 60

Let a, b, c, and d be the lengths of the sides of the quadrilateral, and let p, q, r, and s be the lengths of the segments of the diagonals defined by their intersection (Figure 60). If the diagonals are not perpendicular, let p and r be segments forming an obtuse angle. Then

$$a^2 > p^2 + r^2, \qquad b^2 < r^2 + q^2,$$

$$c^2 > q^2 + s^2, \qquad d^2 < s^2 + p^2.$$

Hence

$$a^2 + c^2 > p^2 + q^2 + r^2 + s^2 > b^2 + d^2.$$

† A plane polygon is *convex* if, for any pair of points of the polygon, the entire line segment connecting them lies in the polygon.

If the diagonals are perpendicular, then by Pythagoras' theorem the inequality signs are to be replaced by equality signs everywhere. Therefore, the diagonals are perpendicular if and only if

$$a^2 + c^2 = b^2 + d^2.$$

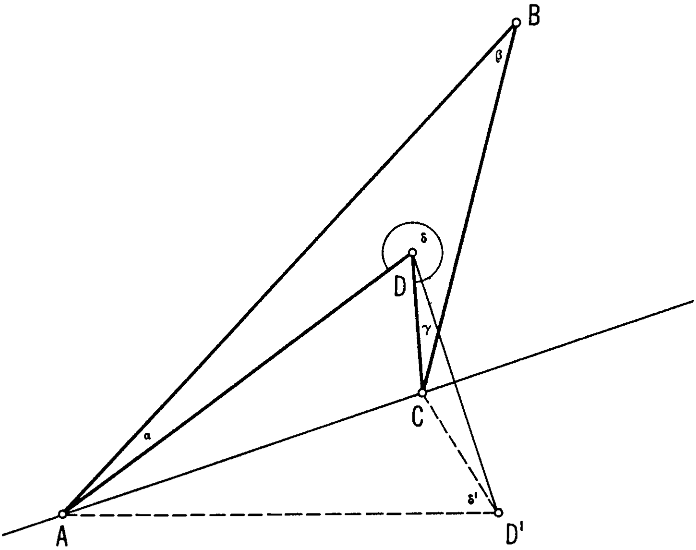

Figure 61

b) If quadrilateral $ABCD$ is not convex and if its inner angles are denoted by α, β, γ, δ, then one of these, say δ, is greater than $180°$ (Figure 61). Reflect the vertex D in the diagonal AC and call the image D'. The sides of quadrilateral $ABCD'$ have the same lengths as those of $ABCD$; the diagonal BD' is perpendicular to the diagonal AC if and only if B lies on line DD', because $DD' \perp AC$ (by definition of a reflection in AC). Hence, $BD' \perp AC$ if and only if $BD \perp AC$. We conclude that the reflection of D in AC affects neither the lengths of the sides, nor the perpendicularity of the diagonals of our quadrilateral.

Since we have solved the problem in the convex case in a), it now suffices to prove that, by a finite number of such reflections of vertices in diagonals, we can always obtain a convex quadrilateral.† To this end observe that in any quadrilateral, there is at most one vertex at which the interior angle is greater than $180°$ (since the sum of all four is $360°$). We now consider the three interior angles α, β, γ of $ABCD$ which are less than $180°$ and compare them with the three interior angles of $ABCD'$ which are less than $180°$. One of these is $\sphericalangle AD'C = \sphericalangle ADC$ since it is equal to $360° - \delta$, and $\delta > 180°$. Another one is $\sphericalangle ABC = \beta$. Since

† Note that after one reflection, the quadrilateral in Figure 61 is still not convex.

either $\sphericalangle BAC$ or $\sphericalangle ACB$ in $\triangle ABC$ is acute, say $\sphericalangle BAC$, we know that $\sphericalangle BAD' < 2 \sphericalangle BAC < 180°$ is another angle of $ABCD'$ which is less than 180°. Now we have that

$$(2) \qquad \begin{gathered} \sphericalangle BAD' > \alpha, \qquad \sphericalangle ABC = \beta, \\ \sphericalangle CD'A = 360° - \delta = \alpha + \beta + \gamma > \gamma; \end{gathered}$$

and adding these, we obtain

$$(3) \quad \sphericalangle BAD' + \sphericalangle ABC + \sphericalangle CD'A > \alpha + \beta + (\alpha + \beta + \gamma).$$

Relation (3) says that, in each step, the sum of the three angles that are less than 180° grows by more than the sum of the two smallest angles of the quadrilateral; and relation (2) says that none of these three angles decreases in size. Therefore, after a finite number of such reflections, the sum of these three angles must exceed 180° and hence the fourth angle must be less than 180°, so that the quadrilateral becomes convex.

Note. Inequalities in triangles. The theorem cited in the Second Solution can be proved by means of the following useful inequality:

If two triangles ABC and $A'B'C'$ have

$$AB = A'B' \quad \textit{and} \quad AC = A'C'$$

then the third side is greater in that triangle in which it lies opposite the greater angle.

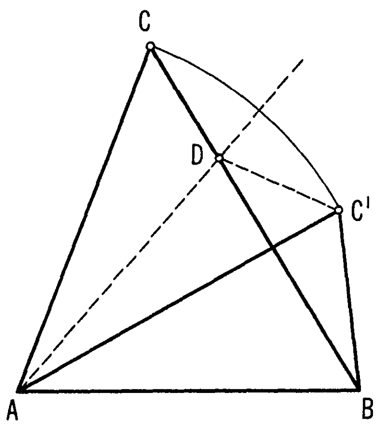

Figure 62

To prove this, let one pair of equal sides coincide so that the other pair has a common vertex. (AB coincides with $A'B'$ in Figure 62 and A is the common vertex of AC and AC'.) Suppose we have labelled the triangles so that

$$\sphericalangle BAC > \sphericalangle BAC'.$$

Let the bisector of $\measuredangle CAC'$ meet BC at D. Since triangles ADC and ADC' are congruent, $DC = DC'$. Therefore

$$BC = BD + DC = BD + DC' > BC',$$

and this proves our contention.

To deduce the theorem of the Second Solution, consider a triangle with sides a, b, c and a right triangle with hypotenuse h and sides b and c. Then, by the above inequality, a is longer or shorter than h depending on whether the angle opposite a is obtuse or acute. An application of Pythagoras' theorem now yields the result.

1913 Competition

1913/1. Prove that for every integer $n > 2$

$$(1 \cdot 2 \cdot 3 \cdots n)^2 > n^n.$$

Solution. The expression on the left in the inequality may be written in the form

$$1 \cdot n \cdot 2(n-1) \cdot 3(n-2) \cdot \cdots \cdot (n-1)2 \cdot n \cdot 1.$$

Now consider the products

$$1 \cdot n, \quad 2 \cdot (n-1), \quad 3 \cdot (n-2), \quad \cdots, \quad (n-1) \cdot 2, \quad n \cdot 1;$$

these are of the form $(k+1)(n-k)$, where k takes on the values $0, 1, 2, \cdots, n-1$. The first product (and the last) is less than the others because, for $n - k > 1$ and $k > 0$,

$$\begin{aligned}(k+1)(n-k) &= k(n-k) + (n-k)\\ &> k \cdot 1 + (n-k) = n.\end{aligned}$$

Now the product of all these products is the expression $(1 \cdot 2 \cdot 3 \cdots n)^2$ and therefore it is greater than $n \cdot n \cdot n \cdots n = n^n$ whenever it has more than two factors, i.e., whenever $n > 2$. This concludes the proof.

1913/2. Let O and O' designate two diagonally opposite vertices of a cube. Bisect those edges of the cube that contain neither of the points O and O'. Prove that these midpoints of edges lie in a plane and form the vertices of a regular hexagon.

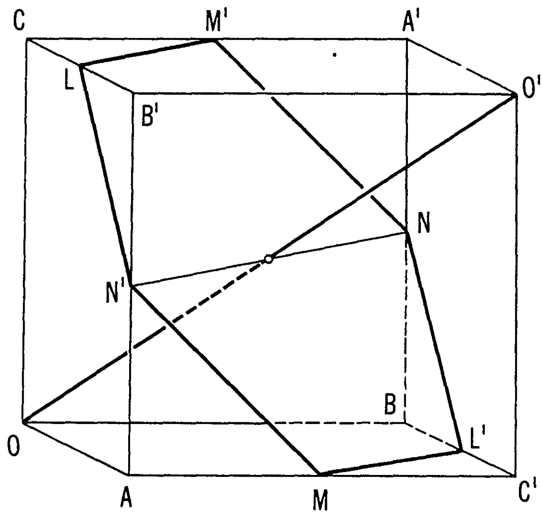

Figure 63

Solution. Let A, B, C (Figure 63) be the endpoints of the three edges originating in O and define A', B', C' similarly with respect to O'. Let

$$L, \quad M', \quad N, \quad L', \quad M, \quad N'$$

be the midpoints, respectively, of

$$B'C, \quad CA', \quad A'B, \quad BC', \quad C'A, \quad AB'.$$

Then the segments

$$O'M, \quad O'N', \quad O'L, \quad O'M', \quad O'N, \quad O'L',$$

$$OM, \quad ON', \quad OL, \quad OM', \quad ON, \quad OL'$$

all have equal lengths because all are hypotenuses of congruent right triangles, since all the faces of the cube are congruent squares. It follows that points M, N', L, M', N, L' lie, simultaneously, on a sphere with center O and on a second sphere with the same radius and center O'. Hence, they lie in the plane perpendicular to the segment OO' and through its midpoint; moreover, they lie on the intersection of the two spheres, hence on a circle.

To show that this inscribed hexagon $MN'LM'NL'$ is regular, we observe that the segment MN' connects the midpoints of sides AB' and AC' of $\triangle AB'C'$ and therefore has half the length of the diagonal $B'C'$ of square $AB'O'C'$, a face of the cube. Since the same is true for segments $N'L$, LM', $M'N$, NL', $L'M$, their lengths are indeed equal and the hexagon is regular.

1913/3. Let d denote the greatest common divisor of the natural numbers a and b, and let d' denote the greatest common divisor of the natural numbers a' and b'. Prove that dd' is the greatest common divisor of the four numbers

$$aa', \quad ab', \quad ba', \quad bb'.$$

First Solution. Define a_1, b_1, a_1', b_1' so that

$$a = a_1 d, \quad b = b_1 d, \quad a' = a_1' d', \quad b' = b_1' d'. \tag{1}$$

Then a_1 and b_1 are relatively prime, and so are a_1' and b_1' [cf. 1901/3, Note 3, (i)]. From (1), we have

$$aa' = dd'a_1a_1', \qquad ab' = dd'a_1b_1',$$

$$ba' = dd'b_1a_1', \qquad bb' = dd'b_1b_1',$$

which shows that dd' is a common divisor of all four numbers. Their greatest common divisor can be greater than dd' only if

$$a_1a_1', \quad a_1b_1', \quad b_1a_1', \quad b_1b_1' \tag{2}$$

have a common prime factor. Suppose p were such a common factor. Since a_1 and b_1 are relatively prime, at most one of them is a multiple of p. Let us say that a_1 is not a multiple of p. But since a_1a_1' is a multiple of p, a_1' is a multiple of p [cf. 1894/1, Second Solution, Note, a)]. Similarly, since a_1b_1' is a multiple of p and a_1 is not, b_1' is a multiple of p. But this contradicts the fact that a_1' and b_1' are relatively prime. Therefore, the numbers (2) have no common divisor greater than 1, and the assertion of the problem is proved.

Second Solution. Let the greatest common divisor of $l, m, \cdots$ be denoted by $(l, m, \cdots)$; then

$$(aa', ab', ba', bb') = \big((aa', ab'), (ba', bb')\big).$$

[See eq. (1) in the Note below.] However,

$$(aa', ab') = a(a', b') = ad',$$

$$(ba', bb') = b(a', b') = bd'.$$

[See eq. (2) in the Note below.] Thus

$$\big((aa', ab'), (ba', bb')\big) = (ad', bd') = (a, b)d',$$

that is,

$$(aa', ab', ba', bb') = dd'$$

as claimed.

Note. Two theorems on the greatest common divisor. The Second Solution is based on the identities

$$(1) \qquad (l, m, \cdots, \quad l', m', \cdots) = ((l, m, \cdots), (l', m', \cdots)),$$

$$(2) \qquad (kl, km, \cdots) = k(l, m, \cdots).$$

Both become obvious from the following circumstance [cf. 1901/3, Note 2]. If the numbers $l, m, \cdots,$ and their greatest common divisor d are written out as products of powers of primes, then the exponent of any prime p in d is the least among the exponents of p in $l, m, \cdots$. To find the smallest exponent of p in all these power products for $l, m, \cdots,$ we may divide our set of numbers into several subsets, find the smallest exponent of p that occurs in each, and then determine the smallest exponent of p among these. This is just what is done in the right-hand side of (1): first the smallest exponent of p in $l, m, \cdots$ is formed, then the smallest exponent of p in $l', m', \cdots,$ and then the smaller of the two is taken.

The identity (2) says that among exponents

$$\alpha + \lambda, \quad \alpha + \mu, \quad \cdots,$$

the smallest is α plus the smallest of the numbers $\lambda, \mu, \cdots$.

1914 Competition

1914/1. Let A and B be points on a circle k. Suppose that an arc k' of another circle l connects A with B and divides the area inside the circle k into two equal parts. Prove that arc k' is longer than the diameter of k.

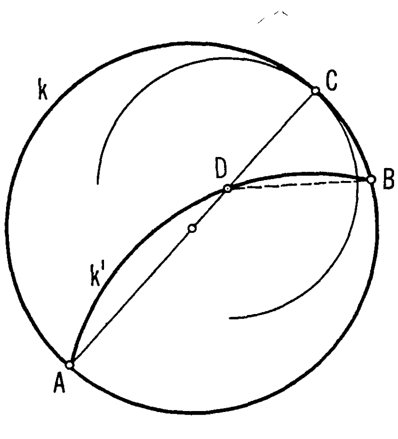

Figure 64

Solution. Since arc k' divides the area of circle k into equal parts, k' cannot lie entirely to one side of any diameter. Hence, every diameter

of k intersects arc k' (Figure 64), and the center O of k lies inside the circle l. Therefore the radius AO of k is inside l, and the intersection D of k' and the diameter AC is on the radius OC.

Since k' is longer than the distance $AD + BD$, we need to show only that $DB > DC$. This is true because the circle through C with center D lies inside the circle k.

1914/2. Suppose that

$$-1 \leq ax^2 + bx + c \leq 1 \qquad \text{for} \quad -1 \leq x \leq 1,$$

where a, b, c are real numbers. Prove that

$$-4 \leq 2ax + b \leq 4 \qquad \text{for} \quad -1 \leq x \leq 1.$$

Solution. The graph of the linear function

$$f'(x) = 2ax + b$$

derived† from the function

$$f(x) = ax^2 + bx + c$$

is a straight line; therefore, as x varies over the interval from -1 to 1, $f'(x)$ assumes its maximum and minimum values at the endpoints of that interval. Hence, it suffices to prove that the values of $f'(x)$ at $x = -1$ and $x = 1$, i.e., the numbers

$$f'(-1) = -2a + b \qquad \text{and} \qquad f'(1) = 2a + b$$

are neither less than -4 nor greater than 4.

To estimate these numbers, we substitute -1, 0, and 1 for x in $f(x) = ax^2 + bx + c$ and use the given inequalities. We obtain

$$(1) \qquad -1 \leq f(-1) = a - b + c \leq 1$$

$$(2) \qquad -1 \leq -f(0) = -c \leq 1$$

$$(3) \qquad -1 \leq f(1) = a + b + c \leq 1.$$

The sum of (1) and (2) gives

$$(4) \qquad -2 \leq a - b \leq 2,$$

† The fact that $f'(x)$ is the derivative of $ax^2 + bx + c$ is not used in this solution; the symbol $f'(x)$ may be interpreted simply as a name for the function $2ax + b$.

the sum of (2) and (3) gives

(5) $$-2 \leq a + b \leq 2,$$

and the sum of (4) and (5) gives

$$-4 \leq (a + b) + (a - b) \leq 4$$

or

(6) $$-2 \leq a \leq 2.$$

Adding first (4) and (6), then (5) and (6), we obtain

$$-4 \leq 2a - b \leq 4, \quad \text{or} \quad -4 \leq -2a + b \leq 4$$

and

$$-4 \leq 2a + b \leq 4,$$

which shows that the extreme values of $f'(x)$ (and hence certainly all its values) in the interval $-1 \leq x \leq 1$ lie within the specified range.

Note. Markov's Theorem on Chebyshev Polynomials. As an example, take the Chebyshev Polynomial†

$$T_2(x) = 2x^2 - 1$$

where $a = 2$, $b = 0$, and $c = -1$, so that

$$T_2'(x) = 4x = 2ax + b.$$

The functions $T_2(x)$ and $T_2'(x)$ are plotted in Figure 65. $T_2(x)$ and $-T_2(x)$ have the following distinguishing feature: Among all quadratic polynomials $f(x)$ with the property

$$-1 \leq f(x) \leq 1 \quad \text{for} \quad -1 \leq x \leq 1,$$

they are the only ones whose derived functions $T_2'(x)$ and $-T_2'(x)$ actually attain the extreme values:

$$T_2'(-1) = -4, \quad T_2'(1) = 4, \quad \text{and} \quad -T_2'(-1) = 4, \quad -T_2'(1) = -4;$$

all other quadratic functions $ax^2 + bx + c$ with the property mentioned in the problem are such that the derived functions satisfy the strict inequality

$$-4 < 2ax + b < 4.$$

This may be seen from the fact that, in view of relations (4), (5), and (6),

$$f'(1) = 2a + b = 4$$

can hold only if $a + b = a - b = a = 2$; then $b = 0$, and from (1) and

† See the discussion of these polynomials in 1899/1, Second Solution and Note.

from (1) and (2) it would follow that $c = -1$. These values of a, b, c yield the function

$$f(x) = 2x^2 - 1 = T_2(x).$$

Similarly $f'(-1) = 4$, $f'(1) = -4$, $f'(-1) = -4$ would determine a, b, and c so that they yield the functions $-T_2(x)$, $-T_2(x)$, and $T_2(x)$.

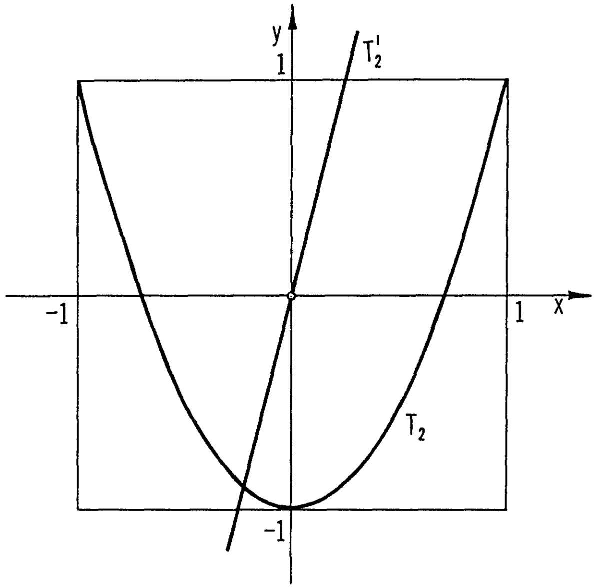

Figure 65

The problem as well as the contents of this Note are special cases of the following theorem due to Markov:†

If a polynomial

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

of degree n, *with real coefficients* $a_0, a_1, \cdots, a_n$, *satisfies*

$$-1 \leq f(x) \leq 1 \quad \textit{for} \quad -1 \leq x \leq 1,$$

then the function

$$f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$$

satisfies

$$-n^2 \leq f'(x) \leq n^2 \quad \textit{for} \quad -1 \leq x \leq 1.$$

Moreover, the equality $|f'(x)| = n^2$ *is attained at most for* $x = -1$ *and* $x = 1$, *and only if* $f(x)$ *is the Chebyshev Polynomial* $T_n(x)$ *or* $-T_n(x)$.

† The Russian mathematician A. Markov was born in 1856 and died in 1922 in St. Petersburg (now Leningrad); he was a university professor and member of the Academy.

1914/3. The circle k intersects the sides BC, CA, AB of triangle ABC in points A_1, A_2; B_1, B_2; C_1, C_2. The perpendiculars to BC, CA, AB through A_1, B_1, C_1, respectively, meet at a point M. Prove that the three perpendiculars to BC, CA, AB through A_2, B_2, and C_2, respectively, also meet in one point.

Solution. The center O of k is on the perpendicular bisector of the segment C_1C_2 (see Figure 66); hence, the perpendicular to AB through C_2 intersects the line through M and O in a point N for which $OM = ON$ (parallel lines cut off proportional segments from the transversals C_1C_2 and MN). Similarly, the perpendicular to BC through A_2 meets the line through M and O in the point N, and so does the perpendicular to AC through B_2. In other words, the perpendiculars through A_2, B_2, C_2 meet in N.

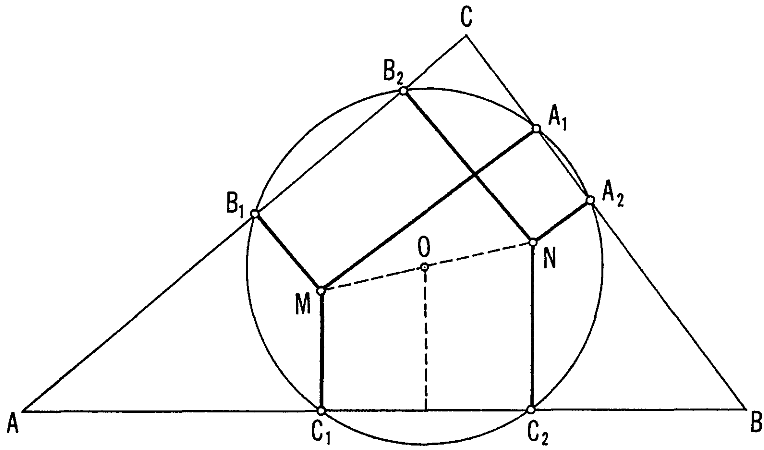

Figure 66

1915 Competition

1915/1. Let A, B, C be any three real numbers. Prove that there exists a number ν such that

$$An^2 + Bn + C < n!$$

for every natural number $n > \nu$.

Solution. Every natural number n satisfies the inequality

$$n! = n(n-1)(n-2)(n-3)\cdots 2 \cdot 1 \geq n(n-1)(n-2).$$

Indeed for $n \geq 4$, $n!$ is the product of the numbers $n(n-1)(n-2)$ and $(n-3)!$, and since $(n-3)! \geq 1$ for $n \geq 4$, the inequality is satisfied in this case; for $n = 1, 2, 3$, we find that $n!$ is, respectively, 1, 2, 6 while $n(n-1)(n-2)$ is, respectively, 0, 0, 6. So the inequality holds for all positive integers n.

Now if we could find a positive integer ν such that, whenever $n > \nu$,

$$An^2 + Bn + C < n(n-1)(n-2), \tag{1}$$

we would have a solution to the problem because then $An^2 + Bn + C$ would certainly be less than $n!$. Let D denote the difference of the right and left members of inequality (1); then (1) is equivalent to the condition

$$D = n(n-1)(n-2) - (An^2 + Bn + C) > 0,$$

which, in turn, is the same as the condition

$$D = n^3 - (Rn^2 + Sn + T) > 0, \tag{2}$$

where $R = A + 3$, $S = B - 2$, $T = C$ are independent of n. Now choose for ν a positive integer greater than either R or S or T; then

$$Rn^2 + Sn + T < \nu(n^2 + n + 1).$$

This together with the fact that

$$n^3 - 1 = (n-1)(n^2 + n + 1) < n^3$$

yields the result

$$\begin{aligned} D &> (n-1)(n^2+n+1) - \nu(n^2+n+1) \\ &= (n-1-\nu)(n^2+n+1), \end{aligned} \tag{3}$$

because we replaced the first term in (2) by something smaller and the second term, which is to be subtracted, by something larger. Relation (3) states that $D > 0$ whenever $n > \nu$.

1915/2. Triangle ABC lies entirely inside a polygon. Prove that the perimeter of triangle ABC is not greater than that of the polygon.

Solution. Extend sides AB, BC, CA beyond points A, B, C, respectively, until they intersect the polygon in points L, M, and N, respectively (Figure 67). The points L, M, N divide the polygon into three polygonal paths which we shall denote by (LM), (MN), and (NL).

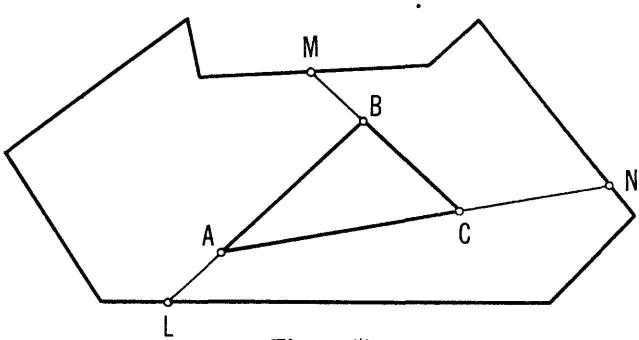

Figure 67

Since a polygonal path is at least as long as the straight line segment connecting its endpoints, we have

$$(LM) + MB \geq LA + AB,$$

$$(MN) + NC \geq MB + BC,$$

$$(NL) + LA \geq NC + CA.$$

Adding these, we obtain

$$(LM) + (MN) + (NL) + MB + NC + LA$$
$$\geq LA + MB + NC + AB + BC + CA$$

or

$$(LM) + (MN) + (NL) \geq AB + BC + CA,$$

which is the desired inequality.

1915/3. Prove that a triangle inscribed in a parallelogram has at most half the area of the parallelogram.

Solution. *First case*: Two vertices of the triangle lie on the same side of the parallelogram; A and B lie on side KL (see Figures 68 and 69). Let t be the area of $\triangle ABC$ and m its altitude from C; let T be the area of parallelogram $KLMN$ and n its altitude to side KL. Then

$$t = \tfrac{1}{2}m \cdot AB, \quad \text{and} \quad T = n \cdot KL.$$

Since

$$m \leq n \quad \text{and} \quad AB \leq KL,$$

it follows that

$$t \leq \tfrac{1}{2}T.$$

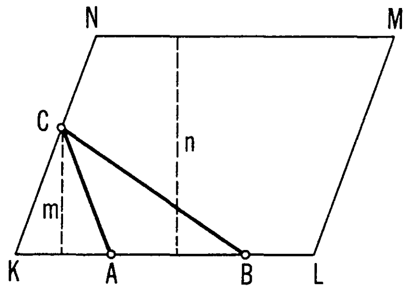

Figure 68

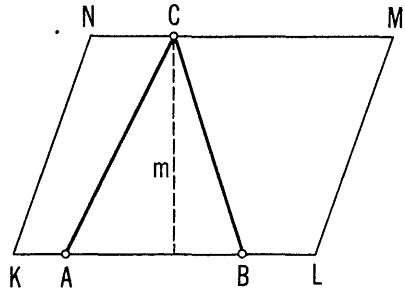

Figure 69

Second case: No two vertices of the triangle lie on the same side of the parallelogram; see Figure 70. In this case two vertices of the triangle must lie on opposite sides of the parallelogram. Suppose A is on KL, B on MN, and C on KN. Draw the line through C parallel to KL; it will intersect AB and LM in points D and E, respectively. By the result derived in the first case, the area of $\triangle ACD$ is not larger than half the area of parallelogram $KLEC$, and that of $\triangle BCD$ is not larger than half the area of parallelogram $MNCE$. Hence, the area of $\triangle ABC$ cannot be larger than half the area of parallelogram $KLMN$.

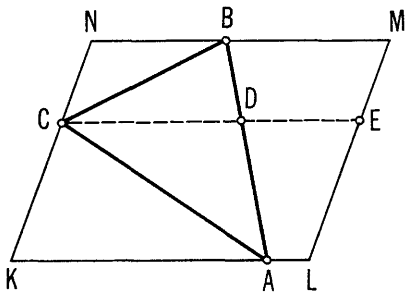

Figure 70

1916 Competition

1916/1. If a and b are positive numbers, prove that the equation

$$\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x+b} = 0$$

has two real roots, one between $a/3$ and $2a/3$, and one between $-2b/3$ and $-b/3$.

Solution. We show first that the equation has two real roots. If we eliminate fractions by multiplying by $x(x - a)(x + b)$, we obtain

$$f(x) = 3x^2 - 2(a - b)x - ab = 0.$$

We find the values of $f(x)$ at $x = -b$, $x = 0$, and $x = a$:

$$\begin{aligned} f(-b) &= 3b^2 - 2(a-b)(-b) - ab = b(a+b) \\ f(0) &= -ab \\ f(a) &= 3a^2 - 2(a-b)a - ab = a(a+b). \end{aligned}$$

Since a and b are positive, this shows that $f(-b)$ is positive, $f(0)$ is negative, and $f(a)$ is positive so that $f(x) = 0$ has a negative root greater than $-b$ and a positive root less than a. These two roots satisfy the original equation because the elimination of fractions could have introduced only the roots $-b$, 0, a. Thus the given equation has two real roots.

If x_1 is the positive root, its substitution in the original equation gives

$$\frac{1}{x_1} + \frac{1}{x_1 + b} = \frac{1}{a - x_1}, \tag{1}$$

and every denominator is positive. Hence

$$\frac{1}{x_1} < \frac{1}{a - x_1}, \quad \text{so that} \quad x_1 > a - x_1 \quad \text{and} \quad x_1 > \frac{a}{2}.$$

Thus x_1 is certainly greater than $a/3$. Moreover, since b is positive,

$$x_1 < x_1 + b, \qquad \frac{1}{x_1} > \frac{1}{x_1 + b},$$

and from (1) it follows that

$$\frac{1}{x_1} + \frac{1}{x_1} > \frac{1}{a - x_1},$$

so that

$$x_1 < \frac{2a}{3}.$$

The negative root x_2 can be treated in the same way and the desired result will then be obtained.

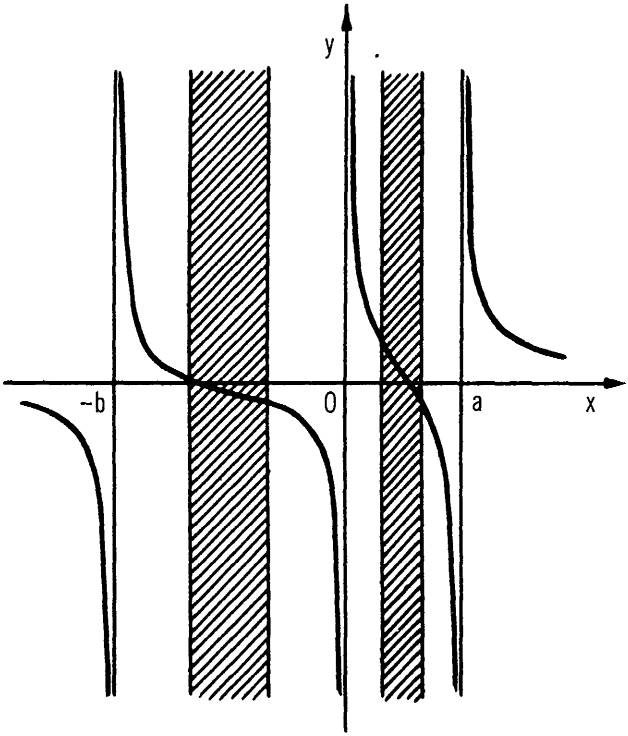

Figure 71

Note. Laguerre's theorem. The equation given in the problem is a special case of

$$\frac{1}{x-a_1}+\frac{1}{x-a_2}+\cdots+\frac{1}{x-a_n}=0 \tag{1}$$

with $a_1, a_2, \cdots, a_n$ distinct real numbers. It has long been known that this equation has $n-1$ real roots and that, if the subscripts are chosen so that $a_1 < a_2 < \cdots < a_n$, then each of the intervals

$$(a_1, a_2), \quad (a_2, a_3), \quad \cdots, \quad (a_{n-1}, a_n)$$

contains exactly one root.

Laguerre† studied the deeper questions: Where, within the interval (a_i, a_{i+1}), is the root x_i located? In particular, can the root x_i be arbitrarily close to one of the endpoints of this interval? This question is answered by

† E. Laguerre (1834–1886), French mathematician, was professor at the *Collège de France* and member of the Academy of Science in Paris. He proved many interesting geometric and algebraic theorems.

the following of his theorems:

If the interval (a_i, a_{i+1}) *is divided into* n *equal segments, then* x_i *is never in the first or last segment.*

For $n = 3$ and $a_1 = -b$, $a_2 = 0$, $a_3 = a$, this theorem becomes the statement of our problem. Figure 71 shows the function

$$f(x) = \frac{1}{x+b} + \frac{1}{x} + \frac{1}{x-a}$$

and the roots of $f(x) = 0$. Our method can be used to prove Laguerre's theorem in its full generality.

1916/2. Let the bisector of the angle at C of triangle ABC intersect side AB in point D. Show that the segment CD is shorter than the geometric mean† of the sides CA and CB.

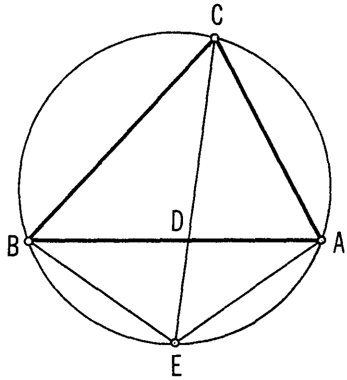

Figure 72

First Solution. Let E be the intersection of CD extended with the circumscribed circle of $\triangle ABC$ (Figure 72). Since $\sphericalangle BCD = \sphericalangle ACD$ and $\sphericalangle CBD = \sphericalangle CEA$,

$$\triangle CBD \sim \triangle CEA$$

and therefore

$$CD:CB = CA:CE \quad \text{or} \quad CA \cdot CB = CD \cdot CE.$$

† The *geometric mean* of two positive numbers is the square root of their product; the geometric mean of n positive numbers is the nth root of their product.

Since CE is longer than CD,

$$CA \cdot CB > (CD)^2;$$

it follows that CD is less than $\sqrt{CA \cdot CB}$, which was to be proved.

Second Solution. a) Denote the lengths of sides BC and AC by a and b and the included angle by γ; let v be the length of the bisector CD of γ. Then, as was shown in 1910/3, First Solution,

$$(1) \qquad \frac{1}{v}\cos\frac{\gamma}{2} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Since $\gamma/2$ is an acute angle, we have

$$0 < \cos\frac{\gamma}{2} < 1,$$

and if we substitute 1 for $\cos(\gamma/2)$ in (1), we increase the left member. Thus

$$(2) \qquad \frac{1}{v} > \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

b) Since the arithmetic mean† of two numbers is not less than their geometric mean (see Note 1 below), that is, since

$$\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \geq \sqrt{\frac{1}{a} \cdot \frac{1}{b}},$$

it follows from (2) that

$$\frac{1}{v} > \sqrt{\frac{1}{a} \cdot \frac{1}{b}},$$

and hence that

$$(3) \qquad v < \sqrt{a \cdot b}.$$

Inequality (2) is *necessary and sufficient* for the existence of a triangle with sides of lengths a and b and with a bisector of the included angle

† The *arithmetic mean* of two numbers is half their sum, also called their average; the arithmetic mean of n numbers is their sum divided by n.

of length v, while (3) is only necessary, but not sufficient. The example

$$a = 3, \quad b = 13, \quad v = 6$$

gives values that satisfy (3):

$$6 < \sqrt{3 \cdot 13} = \sqrt{39};$$

yet no triangle having these parts exists because (2) is not satisfied:

$$\frac{1}{6} \not> \frac{1}{2}\left(\frac{1}{3} + \frac{1}{13}\right) = \frac{8}{39}.$$

Note 1. Comparison of arithmetic and geometric means. *If*

$$x_1, x_2, \cdots, x_n$$

are positive numbers, then their arithmetic mean

$$\frac{x_1 + x_2 + \cdots + x_n}{n}$$

and their geometric mean

$$\sqrt[n]{x_1 x_2 \cdots x_n}$$

satisfy the inequality

$$\text{(1)} \qquad \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

which can also be written

$$\text{(2)} \qquad x_1 x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n;$$

equality holds if and only if

$$x_1 = x_2 = \cdots = x_n.$$

First we shall prove this theorem for $n = 2^k$ by induction on k. To check it for $k = 1$ we must show that

$$\text{(3)} \qquad x_1 x_2 \leq \left(\frac{x_1 + x_2}{2}\right)^2.$$

Now we use the fact that

$$\left(\frac{x_1 + x_2}{2}\right)^2 = \left(\frac{x_1 - x_2}{2}\right)^2 + x_1 x_2$$

or

$$\left(\frac{x_1 + x_2}{2}\right)^2 - x_1x_2 = \left(\frac{x_1 - x_2}{2}\right)^2;$$

and, since we are dealing with real numbers,

$$\left(\frac{x_1 - x_2}{2}\right)^2 \geq 0,$$

where equality holds if and only if $x_1 = x_2$. Therefore

$$\left(\frac{x_1 + x_2}{2}\right)^2 \geq x_1x_2;$$

since x_1 and x_2 are positive,

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1x_2}\,.$$

Let $k = 2$ (i.e., $n = 2^2 = 4$); by using (3) first for x_1, x_2 and then for x_3, x_4 and by multiplying the resulting inequalities, we obtain

$$(x_1x_2)(x_3x_4) \leq \left(\frac{x_1 + x_2}{2}\right)^2 \left(\frac{x_3 + x_4}{2}\right)^2. \tag{4}$$

By using (3) for the numbers

$$\frac{x_1 + x_2}{2} \quad \text{and} \quad \frac{x_3 + x_4}{2},$$

we obtain

$$\left(\frac{x_1 + x_2}{2}\right)\left(\frac{x_3 + x_4}{2}\right) \leq \left[\frac{1}{2}\left(\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}\right)\right]^2 = \left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right)^2;$$

squaring both sides yields

$$\left(\frac{x_1 + x_2}{2}\right)^2 \left(\frac{x_3 + x_4}{2}\right)^2 \leq \left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right)^4$$

which, together with (4), yields the desired result

$$x_1x_2x_3x_4 \leq \left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right)^4.$$

Equality holds if and only if it held in all the relations used to derive this one, that is, if and only if $x_1 = x_2 = x_3 = x_4$.

If $k = 3$ $(n = 2^3 = 8)$, we use the inequalities

$$(x_1x_2x_3x_4)(x_5x_6x_7x_8) \le \left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right)^4 \left(\frac{x_5 + x_6 + x_7 + x_8}{4}\right)^4,$$

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right)\left(\frac{x_5 + x_6 + x_7 + x_8}{4}\right)$$

$$\le \left(\frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8}{8}\right)^2$$

and obtain, as before,

$$x_1x_2x_3x_4x_5x_6x_7x_8 \le \left(\frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8}{8}\right)^8.$$

By continuing in this way, we can verify the theorem for $4, 8, 16, \cdots, 2^k$ positive numbers.

Finally, for arbitrary n, we proceed as follows: If, for example, $n = 5$, we add to our n numbers $8 - 5 = 3$ more numbers, each equal to the arithmetic mean m of the 5 given numbers. Since 8 is a power of 2, we already know that

$$x_1x_2x_3x_4x_5m^3 \le \left(\frac{x_1 + x_2 + x_3 + x_4 + x_5 + 3m}{8}\right)^8 = m^8,$$

whence

$$x_1x_2x_3x_4x_5 \le m^5.$$

This surprising method of proof is due to Cauchy.†

If $x_1, x_2, \cdots, x_n$ are variables but their sum or their product is *constant*, then the theorem implies the following assertions:

If the factors in a product vary over the positive numbers in such a way that their sum remains constant, then the product is greatest when all factors are equal.

If the summands in a sum vary over the positive numbers in such a way that their product remains constant, then the sum is least when all the summands are equal.

† The French mathematician A. L. Cauchy was born in Paris in 1789 and died there in 1857. He was a member of the Academy of Science and professor at the *École Polytechnique.* The theorem cited above is taken from his *Cours d'Analyse* (1821). This basic treatise contains the first rigorous interpretations of limit and the first rigorous treatment of infinite series. Another great accomplishment of Cauchy is the founding of the theory of functions of a complex variable.

Note 2. Jensen's theorem. Following Cauchy's method of proof step by step, Jensen† found the theorem:

If a function $f(x)$ *has the property that, for any pair of real values* x_1, x_2 *in some (possibly infinite) interval of the real line*

$$(1) \qquad f(x_1) + f(x_2) \leq 2f\left(\frac{x_1 + x_2}{2}\right),$$

and if $x_1, x_2, \cdots, x_n$ *belong to the same interval, then*

$$(2) \qquad f(x_1) + f(x_2) + \cdots + f(x_n) \leq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

[*If in* (1) *equality holds if and only if* $x_1 = x_2$, *then equality holds in* (2) *if and only if* $x_1 = x_2 = \cdots = x_n$.]

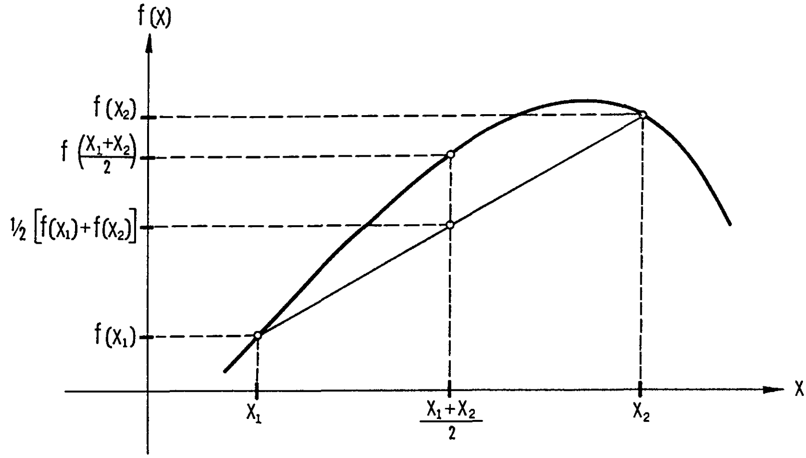

Figure 73

Before sketching the proof we observe that inequality (1), the hypothesis of Jensen's theorem, has the following geometric meaning (Figure 73): the midpoint of the chord connecting any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ of the graph of the function does not lie above the value of the function taken at the average $\frac{1}{2}(x_1 + x_2)$ of x_1 and x_2. This is certainly true if the graph of a function is nowhere below any chord connecting two of its points. A function with this property is called *convex downward.* (In the opposite case, where the graph nowhere lies above its chords, the function is called *convex upward*; see Figure 74.)

† The Danish mathematician J. L. V. W. Jensen (1859–1925) was managing chief engineer of the telephone network of Denmark and member of the Academy of Science.

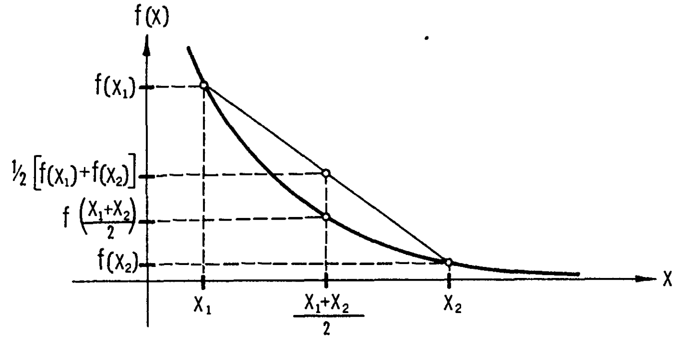

Figure 74

To prove Jensen's theorem for $n = 2^2 = 4$, we write inequality (1) for x_1 and x_2, then for x_3 and x_4, then add these and apply (1) again to the result:

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) \leq 2\left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right)\right]$$

$$\leq 2\left[2f\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right)\right].$$

It is clear how one achieves the proof for $n = 2^k$.

For arbitrary n, we again follow Cauchy's method and add $2^k - n$ additional terms:

$$(3) \quad f(x_1) + f(x_2) + \cdots + f(x_n) + (2^k - n)f(x')$$
$$\leq 2^k f\left(\frac{x_1 + x_2 + \cdots + x_n + (2^k - n)x'}{2^k}\right),$$

where

$$x' = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

We observe that the argument of f in the right member of (3) can be written

$$\frac{x_1 + x_2 + \cdots + x_n + 2^k x' - (x_1 + x_2 + \cdots + x_n)}{2^k} = x'$$

so that (3) becomes

$$f(x_1) + f(x_2) + \cdots + f(x_n) + 2^k f(x') - nf(x') \leq 2^k f(x'),$$

whence we derive the desired result

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

Jensen's theorem contains the theorem of Note 1 as a special case (provided it has been verified for $n = 2$); it also affords new solutions for two of our previous problems.

a) *Application to the inequality of the geometric and arithmetic means.* If x_1 and x_2 are positive, then we have shown that

$$x_1x_2 \leq \left(\frac{x_1 + x_2}{2}\right)^2.$$

It follows that

$$\log x_1 + \log x_2 \leq 2 \log \frac{x_1 + x_2}{2}.$$

Jensen's theorem then yields the inequality

$$\log x_1 + \log x_2 + \cdots + \log x_n \leq n \log \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for any positive numbers $x_1, x_2, \cdots, x_n$ and this, in return, implies the inequality (2) of Note 1.

b) *Application to 1897/2.* Let x_1 and x_2 be positive acute angles. From the identities

$$\sin x_1 \sin x_2 = \tfrac{1}{2}[\cos(x_1 - x_2) - \cos(x_1 + x_2)],$$

$$\sin^2\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}[1 - \cos(x_1 + x_2)],$$

(derived in 1897/1, Note), we obtain

$$\sin^2\left(\frac{x_1 + x_2}{2}\right) - \sin x_1 \sin x_2 = \frac{1}{2}[1 - \cos(x_1 - x_2)] = \sin^2\left(\frac{x_1 - x_2}{2}\right),$$

and hence

$$\sin x_1 \sin x_2 = \sin^2\left(\frac{x_1 + x_2}{2}\right) - \sin^2\left(\frac{x_1 - x_2}{2}\right)$$

so that

$$\sin x_1 \sin x_2 \leq \sin^2\left(\frac{x_1 + x_2}{2}\right)$$

and

$$\log \sin x_1 + \log \sin x_2 \leq 2 \log \sin\left(\frac{x_1 + x_2}{2}\right).$$

By Jensen's theorem

$$\log \sin x_1 + \log \sin x_2 + \cdots + \log \sin x_n \leq n \log \sin \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for any set of n positive acute angles $x_1, x_2, \cdots, x_n$. It follows that

$$\sin x_1 \sin x_2 \cdots \sin x_n \leq \sin^n \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \tag{4}$$

with equality only for $x_1 = x_2 = \cdots = x_n$.

If $n = 3$ and $x_1 + x_2 + x_3 = 90°$ so that x_1, x_2, x_3 are half angles of a triangle, then inequality (4) becomes

$$\sin x_1 \sin x_2 \sin x_3 \leq \sin^3 30° = \tfrac{1}{8}.$$

This is the inequality of 1897/2, Second Solution b), obtained there by Euler's formula.

c) *Application to 1898/2.* If x_1, x_2 are angles between $0°$ and $180°$, then

$$\sin x_1 + \sin x_2 = 2 \sin \frac{x_1 + x_2}{4} \cos \frac{x_1 - x_2}{2} \leq 2 \sin \frac{x_1 + x_2}{2}.$$

By Jensen's theorem,

$$\sin x_1 + \sin x_2 + \cdots + \sin x_n \leq n \sin \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for any n angles between $0°$ and $180°$, with equality if and only if

$$x_1 = x_2 = \cdots = x_n.$$

In particular, for the angles in a triangle we have $x_1 + x_2 + x_3 = 180°$ and the inequality yields the solution of the second part of 1898/2.†

Observe that in all three applications given here, the major part of the work consisted in verifying the hypothesis of Jensen's theorem; this may be done by showing that the function is convex. It can be shown that the convexity (downward) of $f(x)$ in the interval $a < x < b$ means that the second derivative, $f''(x)$, of $f(x)$ is non-positive. This condition is easily verified in the above applications.

In a),

$$f(x) = \log x, \qquad f'(x) = \frac{1}{x},$$

$$f''(x) = -\frac{1}{x^2} < 0 \quad \text{for} \quad x > 0.$$

† The following remarks are addressed mainly to readers familiar with the calculus.

In b),

$$f(x) = \log \sin x, \qquad f'(x) = \frac{\cos x}{\sin x},$$

$$f''(x) = \frac{-\cos^2 x - \sin^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} < 0 \quad \text{for} \quad 0 < x < 180°.$$

Finally, in c),

$$f(x) = \sin x, \qquad f'(x) = \cos x,$$

$$f''(x) = -\sin x < 0 \quad \text{for} \quad 0 < x < 180°.$$

1916/3. Divide the numbers

$$1, \quad 2, \quad 3, \quad 4, \quad 5$$

into two arbitrarily chosen sets. Prove that one of the sets contains two numbers and their difference.

Solution. Let us try to divide $1, 2, 3, 4, 5$ into two sets in such a way that neither set contains the difference of two of its members.

2 cannot be in the same set with 1 or 4 because $2 - 1 = 1$, $4 - 2 = 2$; so let us put 2 in one set, 1 and 4 in the second. Then we may not put 3 in the second, since $4 - 1 = 3$; so let us put 3 in the first. But then 5 cannot be put either in the first ($5 - 3 = 2$, $5 - 2 = 3$), or in the second ($5 - 4 = 1$, $5 - 1 = 4$).

1917 Competition

1917/1. If a and b are integers and if the solutions of the system of equations

$$y - 2x - a = 0 \tag{1}$$

$$y^2 - xy + x^2 - b = 0 \tag{2}$$

are rational, prove that the solutions are integers.

Solution. Equation (1) gives

$$x = \frac{y - a}{2};$$

when this is substituted in (2) one obtains

$$y^2 - y\left(\frac{y - a}{2}\right) + \left(\frac{y - a}{2}\right)^2 - b = 0. \tag{2'}$$

When this is simplified, it becomes

$$3y^2 = 4b - a^2 \tag{2''}$$

and when multiplied by 3, it becomes

$$(3y)^2 = 3(4b - a^2). \tag{3}$$

Any rational numbers x and y that satisfy (1) and (2) necessarily also satisfy (3).

The integer $3(4b - a^2)$ on the right side of (3) can be the square of $3y$ only if $3y$ is an integer (cf. 1907/1, Solution and Note). Moreover, the right side of (3) is divisible by 3; therefore the integer $3y$ is also divisible by 3 and so y is indeed an integer (cf. 1894/1, Second Solution, Note).

We see from (2″) that a and y are either both even or both odd. For, if y is odd, $3y^2$ is odd and $4b - a^2$ can be odd only if a^2 is odd in which case a is odd; similarly, if y is even, a is even. It follows that $y - a$ is even in any case so that

$$x = \frac{y - a}{2}$$

is an integer.

1917/2. In the square of an integer a, the tens' digit is 7. What is the units' digit of a^2 ?

Solution. We shall show that the units' digit of a^2 is 6 whenever its tens' digit is odd.

Let c be the units' digit of a. Then clearly, $a + c$ is even and $a - c$ is divisible by 10. Consequently, $a^2 - c^2 = (a + c)(a - c)$ is divisible by 20. It follows that a^2 and c^2 have equal units' digits and that their tens' digits are either both even or both odd. So, whenever the tens'

digit of a^2 is odd, c must be either 4 or 6 because the squares of all other one-digit integers have even tens' digits. Since the squares of 4 and of 6 have the units' digit 6, it follows that the units' digit of a^2 is 6.

1917/3. Let A and B be two points inside a given circle k. Prove that there exist (infinitely many) circles through A and B which lie entirely in k.

Solution. Connect A and B with the center O of the circle k. If P is any point on OA (or on OB) then a circle about P with radius PA (respectively PB) lies inside k. The perpendicular bisector of AB (Figure 75) intersects either OA or OB. If C is the point of intersection, the circle about C with radius $CA = CB$ lies inside k.

The perpendicular bisector of AB is the locus of centers of circles through A and B and contains other points, near C, which would serve as centers of circles that lie inside k and go through A and B.

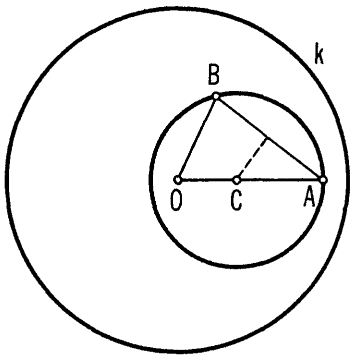

Figure 75

1918 Competition

1918/1. Let AC be the longer of the two diagonals of the parallelogram $ABCD$. Drop perpendiculars from C to AB and AD extended. If E and F are the feet of these perpendiculars, prove that

$$AB \cdot AE + AD \cdot AF = (AC)^2.$$

Solution. Let G be the foot of the perpendicular to AC through B (Figure 76). Since AC divides $ABCD$ into two obtuse triangles,

G does not coincide with either A or C, but lies on the segment AC. The right triangles AEC and AGB are similar because they share the angle BAC. Also, right triangles AFC and CGB are similar because angle CAD is equal to angle ACB. Hence

$$\frac{AC}{AE} = \frac{AB}{AG} \quad \text{and} \quad \frac{AC}{AF} = \frac{BC}{GC}$$

so that

$$AB \cdot AE = AC \cdot AG, \qquad BC \cdot AF = AC \cdot GC$$

and

$$AB \cdot AE + BC \cdot AF = AC(AG + GC).$$

However,

$$BC = AD, \qquad AG + GC = AC,$$

and substitution in the relation above yields

$$AB \cdot AE + AD \cdot AF = AC^2$$

as we set out to prove.

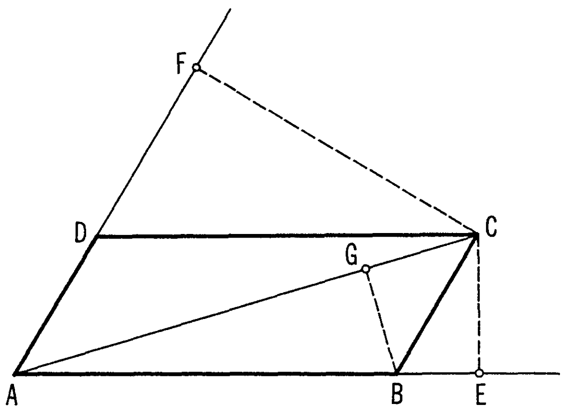

Figure 76

This formula implies Pythagoras'† theorem as a special case: If $ABCD$ is a rectangle, the formula becomes

$$(AB)^2 + (BC)^2 = (AC)^2.$$

† Pythagoras, Greek philosopher of the sixth century B. C., brought some of his philosophical ideas from Egypt. He lived in southern Italy and founded a school in the city of Crotona.

1918/2. Find three distinct natural numbers such that the sum of their reciprocals is an integer.

Solution. We seek positive integers x, y, z such that

$$x < y < z, \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a,$$

where a is an integer. Since x, y, z cannot be smaller than 1, 2, 3, respectively, we have

$$\begin{aligned} a &= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \\ &\leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \\ &< \frac{1}{1} + \frac{1}{2} + \frac{1}{2} \\ &= 2 \end{aligned}$$

so that $a = 1$. Since $x < y < z$, we have $1/x > 1/y > 1/z$, and we see that

$$\begin{aligned} \frac{1}{x} < a &= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \\ &< \frac{1}{x} + \frac{1}{x} + \frac{1}{x} = \frac{3}{x}; \end{aligned}$$

hence

$$1 < x < 3, \quad \text{so} \quad x = 2, \quad \text{and} \quad \frac{1}{y} + \frac{1}{z} = \frac{1}{2}.$$

This and $1/z < 1/y$ give the inequality

$$\frac{1}{y} < \frac{1}{2} < \frac{2}{y}$$

which, when multiplied by $2y$ becomes

$$2 < y < 4, \quad \text{so} \quad y = 3, \quad \text{and} \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{z} = 1.$$

Solving for z we obtain $z = 6$. The solution is

$$x = 2, \quad y = 3, \quad z = 6,$$

and the sum of their reciprocals is the integer 1.

1918/3. If $a, b, c; p, q, r$ are real numbers such that, for every real number x,

$$ax^2 + 2bx + c \geq 0 \quad \text{and} \quad px^2 + 2qx + r \geq 0,$$

prove that then

$$apx^2 + 2bqx + cr \geq 0$$

for all real x.

Solution. a) *Lemma. If for every real x*

$$(1) \qquad f(x) = ax^2 + 2bx + c \geq 0,$$

then

$$a, \quad c, \quad \textit{and} \quad ac - b^2$$

are non-negative.

The proof is based on the identity

$$(2) \qquad af(x) = (ax + b)^2 + ac - b^2 = (ax + b)^2 - (b^2 - ac).$$

(i) We first prove that $a \geq 0$. If a were negative then, according to (1), $af(x) \leq 0$ for all x; hence, by (2),

$$af(x) = (ax + b)^2 - (b^2 - ac) \leq 0$$

or

$$(3) \qquad (ax + b)^2 \leq b^2 - ac \qquad \text{for all } x.$$

But this cannot be true for some values of x; for example, find a number M which is greater than 1 and also greater than $b^2 - ac$, and let x be the solution of

$$ax + b = M;$$

then

$$(ax + b)^2 = M^2 > M > b^2 - ac,$$

and this contradicts (3); so, a is not negative.

(ii) Next we show that $c \geq 0$. Since $f(x)$ is never negative we have, in particular, for $x = 0$,

$$f(0) = c \geq 0.$$

(iii) We prove that $ac - b^2 \geq 0$ by considering first the case $a > 0$, and then the case $a = 0$.

If $a > 0$ and if x is the solution of

$$ax + b = 0,$$

then, since $f(x)$ is not negative, (2) yields

$$af(x) = ac - b^2 \geq 0.$$

If $a = 0$, (1) becomes

$$f(x) = 2bx + c.$$

Since $f(x) \geq 0$ for all x, b must be zero [otherwise $f(x)$ would be negative for all $x < -c/2b$]. Thus, since a and c are non-negative, we have $ac - b^2 \geq 0$. The result $ac - b^2 \geq 0$ holds in the case $a > 0$ as well as in the case $a = 0$.

b) *Converse of the Lemma. If*

$$a \geq 0, \quad c \geq 0, \quad ac - b^2 \geq 0,$$

then

$$f(x) = ax^2 + 2bx + c \geq 0$$

for all real values of x.

The identity

$$af(x) = (ax + b)^2 + ac - b^2$$

tells us that

$$af(x) \geq ac - b^2$$

since the square of $(ax + b)$ is never negative for real x. It follows that

$$af(x) \geq 0.$$

If $a > 0$, this implies that $f(x) \geq 0$. If $a = 0$, $ac - b^2 \geq 0$ implies that $b = 0$. In this case, $f(x) = c$ for all x and so, under our assumptions, $f(x) \geq 0$.

c) *Solution of the original problem.* If

$$ax^2 + 2bx + c \geq 0, \qquad px^2 + 2qx + r \geq 0$$

for all real x, then, by the lemma,

$$a \geq 0, \quad c \geq 0, \quad p \geq 0, \quad r \geq 0,$$

and

$$ac - b^2 \geq 0, \quad pr - q^2 \geq 0;$$

or,

$$ac \geq b^2, \quad pr \geq q^2.$$

These imply

$$ap \geq 0, \quad cr \geq 0, \quad ac \cdot pr \geq b^2q^2, \quad ap \cdot cr - (bq)^2 \geq 0.$$

According to the converse of the lemma, these inequalities imply that

$$apx^2 + 2bqx + cr \geq 0$$

for all x.

1922 Competition

1922/1. Given four points A, B, C, D in space, find a plane S equidistant from all four points and having A and C on one side, B and D on the other.

Solution. The plane S (Figure 77) is to be equidistant from A and B, and is to have A on one side of it and B on the other. The same has to be true for the pairs of points B, C; C, D; D, A.

Two points on different sides of a plane are equidistant from the plane if and only if the midpoint of the segment connecting them is on the plane. Thus we must show that the midpoints P, Q, R, and T of AB, BC, CD and DA all lie in one plane. Let S be the plane determined by the three points P, Q and R. In triangle ABC, the segment PQ connecting the midpoints of two sides is parallel to the third side AC. Similarly, in $\triangle ADC$, $TR \parallel AC$. Hence $TR \parallel PQ$ and T is in the same plane as the points P, Q and R.

If the midpoints of the segments AB, BC and CD are collinear, every plane through them has the required property.

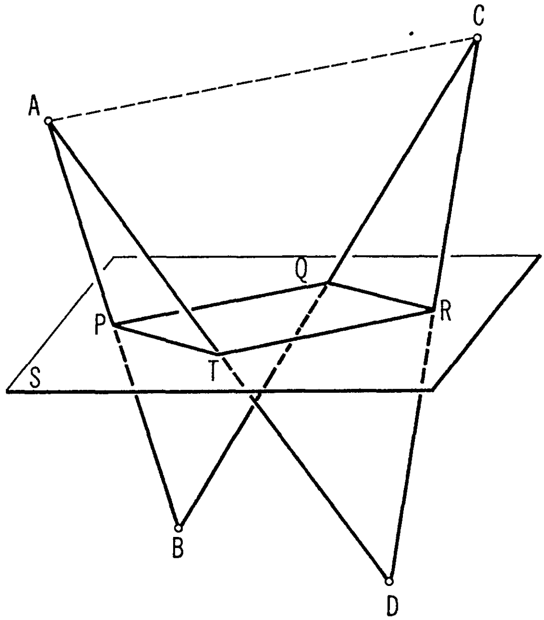

Figure 77

1922/2. Prove that

$$x^4 + 2x^2 + 2x + 2$$

is not the product of two polynomials

$$x^2 + ax + b \quad \text{and} \quad x^2 + cx + d$$

in which a, b, c, d are integers.

Solution. The product $(x^2 + ax + b)(x^2 + cx + d)$ equals

$$x^4 + (a + c)x^3 + (b + ac + d)x^2 + (bc + ad)x + bd.$$

This expression is equal to $x^4 + 2x^2 + 2x + 2$ if and only if

(1) $$a + c = 0$$

(2) $$b + ac + d = 2$$

(3) $$bc + ad = 2$$

(4) $$bd = 2.$$

Since a, b, c, d are to be integers, one of the factors in (4) must be odd (± 1), the other even (± 2). Suppose b is odd and d is even; then

(3) implies that bc is even, hence that c is even. Since b is odd and d, c are even, the left side of (2) is odd. This is impossible since the right side is even. Similarly, the assumption that b is even and d is odd would have led to a contradiction. Hence the given fourth degree polynomial cannot be factored into two quadratic polynomials with integer coefficients.

Note. Eisenstein's Theorem. The theorem just proved is a special case of the following: *If*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

has integral coefficients such that $a_1, a_2, \cdots, a_n$ *are multiples of a prime* p *while* $a_0 \neq 0$ *is prime to* p, *and* a_n *is not divisible by* p^2, *then* $f(x)$ *cannot be split into a product of polynomials of degree lower than n with integral coefficients.* This theorem on irreducible polynomials is due to Schoenemann (1846) and Eisenstein (1850); it is usually credited to Eisenstein.†

1922/3. Show that, if $a, b, \cdots, n$ are distinct natural numbers, none divisible by any primes greater than 3, then

$$\frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{n} < 3.$$

Solution. By assumption, each term in

$$S = \frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{n}$$

is of the form

$$\frac{1}{2^r3^s},$$

where r and s are non-negative integers (cf. 1896/1, Note). Let t be the largest exponent that occurs among these r and s. Then every term of S is the product of some term in the geometric progression

$$U = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^t}$$

† The German mathematician and physicist T. Schoenemann was born in 1812; he was a high school teacher in Brandenburg and died there in 1868.

F. G. M. Eisenstein (Berlin, 1823–1852) was professor at the University of Breslau.

with some term in the geometric progression

$$V = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^t};$$

in other words, each term in S is a term in the expression UV. Since $a, b, \cdots, n$ are distinct, different terms of S correspond to distinct terms of UV. Hence $S \leq UV$. We shall now show that $U < 2$, $V < \frac{3}{2}$, from which it will follow that $S < 3$:

$$U = \frac{1 - (1/2)^{t+1}}{1 - (1/2)} < \frac{1}{1 - (1/2)} = 2$$

$$V = \frac{1 - (1/3)^{t+1}}{1 - (1/3)} < \frac{1}{1 - (1/3)} = \frac{3}{2}.$$

This completes the proof.

1923 Competition

1923/1. Three circles through the point O and of radius r intersect pairwise in the additional points A, B, C. Prove that the circle through the points A, B, and C also has radius r.

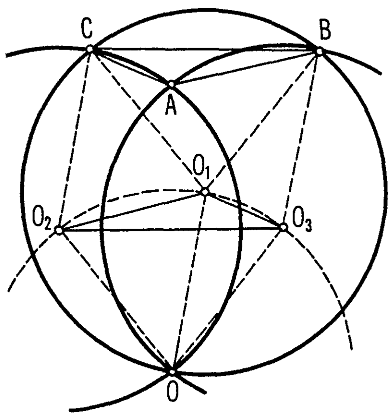

Figure 78

Solution. The centers O_1, O_2, O_3 of the given circles OBC, OCA, and OAB (Figure 78) have distance r from O. Hence, the circle circumscribed about triangle $O_1O_2O_3$ has radius r. If we can prove that

$$\triangle ABC \cong \triangle O_1O_2O_3,$$

then we can deduce that the circle through A, B, and C also has radius r. To establish the congruence of these triangles, we observe that the quadrilaterals BO_3OO_1 and CO_2OO_1 are rhombuses since all their sides have the length r. Hence O_3B and O_2C are parallel and equal to OO_1, consequently parallel and equal to each other. This shows that BCO_2O_3 is a parallelogram and therefore

$$BC = O_2O_3.$$

By using the rhombuses AO_3OO_2 and BO_1OO_3 we may prove, in an analogous manner, that

$$AB = O_1O_2 \quad \text{and similarly that} \quad AC = O_1O_3.$$

Thus the desired congruence of triangles ABC and $O_1O_2O_3$ is established.

1923/2. If

$$s_n = 1 + q + q^2 + \cdots + q^n,$$

and

$$S_n = 1 + \frac{1+q}{2} + \left(\frac{1+q}{2}\right)^2 + \cdots + \left(\frac{1+q}{2}\right)^n,$$

prove that

$$(1) \quad \binom{n+1}{1} + \binom{n+1}{2} s_1 + \binom{n+1}{3} s_2 + \cdots + \binom{n+1}{n+1} s_n = 2^n S_n.$$

First Solution. It is sufficient to prove that the coefficients of q^k ($k = 0, 1, 2, \cdots, n$) on each side of (1) are equal. On the left side of (1) q^k occurs with a non-zero coefficient in the terms involving those s_i for which $i \geq k$. Hence, the coefficient of q^k on the left is

$$(2) \qquad \binom{n+1}{k+1} + \binom{n+1}{k+2} + \cdots + \binom{n+1}{n+1}.$$

To determine the coefficients of q^k on the right of (1), imagine each term

$$\left(\frac{1+q}{2}\right)^j \quad \text{with exponent } j \geq k,$$

expanded by the binomial theorem (see Note 2 to 1902/1, NML 11, p. 84):

$$\left(\frac{1+q}{2}\right)^j = \frac{1}{2^j}(q+1)^j$$

$$= \frac{1}{2^j}\left[\binom{j}{j}q^j + \binom{j}{j-1}q^{j-1} + \cdots + \binom{j}{1}q + 1\right].$$

Now let $j = k$, then $j = k+1$, $\cdots$, $j = n$ and collect in each resulting expression the coefficient of q^k:

$$\frac{1}{2^k}\binom{k}{k} + \frac{1}{2^{k+1}}\binom{k+1}{k} + \cdots + \frac{1}{2^n}\binom{n}{k}.$$

Since the right side of (1) contains the factor 2^n, the desired coefficient of q^k on the right of (1) is

$$(3) \qquad 2^{n-k}\binom{k}{k} + 2^{n-k-1}\binom{k+1}{k} + \cdots + \binom{n}{k}.$$

We must now prove that the expression (2) is equal to the expression (3). We shall do this by means of an induction on $n-k$. The first step consists of verifying that (2) and (3) are equal when $n-k=0$, i.e., when $n=k$; in this case (2) reduces to

$$\binom{k+1}{k+1} = 1 \qquad \text{and (3) to} \qquad 2^0\binom{k}{k} = 1,$$

so that (2) and (3) are indeed equal.

Next, assume that (2) and (3) are equal for each of the values $n-k = 0, 1, 2, \cdots, m-1$, i.e. that

$$(4) \quad \binom{n+1}{k+1} + \binom{n+1}{k+2} + \cdots + \binom{n+1}{n+1}$$

$$= 2^{n-k}\binom{k}{k} + 2^{n-k-1}\binom{k+1}{k} + \cdots + \binom{n}{k}$$

for $n-k < m$. We shall show that then (4) holds also for $n-k = m$. For this purpose, it is best to rewrite the left side of (4) using the identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

[cf. 1895/1, Note 2, d)]. Then we obtain

$$\left[\binom{n}{k}+\binom{n}{k+1}\right]+\left[\binom{n}{k+1}+\binom{n}{k+2}\right]$$

$$+\cdots+\left[\binom{n}{n-1}+\binom{n}{n}\right]+\binom{n}{n}$$

or

$$(2') \qquad \binom{n}{k}+2\left[\binom{n}{k+1}+\binom{n}{k+2}+\cdots+\binom{n}{n}\right].$$

Observe that our induction hypothesis is applicable to the expression in brackets since $n-(k+1)=(n-1)-k<m$. We replace n by $n-1$ in (4) and obtain

$$\binom{n}{k+1}+\binom{n}{k+2}+\cdots+\binom{n}{n}$$

$$=2^{n-1-k}\binom{k}{k}+2^{n-2-k}\binom{k+1}{k}+\cdots+\binom{n-1}{k}.$$

We now multiply both sides of this relation by 2 and add

$$\binom{n}{k}$$

to both sides. The result is

$$\binom{n}{k}+2\left[\binom{n}{k+1}+\binom{n}{k+2}+\cdots+\binom{n}{n}\right]$$

$$=2^{n-k}\binom{k}{k}+2^{n-1-k}\binom{k+1}{k}+\cdots+\binom{n-1}{k}+\binom{n}{k};$$

since the left side is just $(2')$ which is equal to (2), and the right side is (3), we have established the equality of (2) and (3) for $n-k=m$. It now follows that (2) and (3) are equal for all $n \geq k$, i.e., that the coefficients of q^k on each side of (1) are identical so that (1) is indeed an identity.

Second Solution. We shall prove the identity (1) first for $q \neq 1$, and then for $q = 1$.

If $q \neq 1$, the formula for a finite geometric series yields

$$s_n = \frac{1 - q^{n+1}}{1 - q}, \tag{2}$$

$$S_n = \frac{1 - \left(\frac{1+q}{2}\right)^{n+1}}{1 - \frac{1+q}{2}} = 2\,\frac{1 - \left(\frac{1+q}{2}\right)^{n+1}}{1 - q}. \tag{3}$$

When we multiply the left side of (1) by $1 - q$ and observe, from (2), that $(1 - q)s_k = 1 - q^{k+1}$, we obtain

$$\binom{n+1}{1}(1 - q) + \binom{n+1}{2}(1 - q^2) + \binom{n+1}{3}(1 - q^3) + \cdots + \binom{n+1}{n+1}(1 - q^{n+1}). \tag{4}$$

Since $1 - q^0 = 0$, (4) will not change its value if we add the term

$$\binom{n+1}{0}(1 - q^0)$$

as first term. After this modification, we write (4) as the difference

$$\left[\binom{n+1}{0}\cdot 1 + \binom{n+1}{1}\cdot 1 + \cdots + \binom{n+1}{n+1}\cdot 1\right] - \left[\binom{n+1}{0}q^0 + \binom{n+1}{1}q + \cdots + \binom{n+1}{n+1}q^{n+1}\right]$$

and recognize the first bracket as the binomial expansion of $(1 + 1)^{n+1}$, and the second as the binomial expansion of $(1 + q)^{n+1}$. So far, we have shown that the left member of (1) multiplied by $(1 - q)$ is equal to

$$(1 + 1)^{n+1} - (1 + q)^{n+1} = 2^{n+1} - (1 + q)^{n+1} = 2^{n+1}\left[1 - \left(\frac{1+q}{2}\right)^{n+1}\right].$$

If we now multiply the right member of (1) by $(1 - q)$ and use the expression derived in (3) for S_n, we obtain

$$2^n(1-q)S_n = 2^n \cdot 2\left[1 - \left(\frac{1+q}{2}\right)^{n+1}\right]$$

$$= 2^{n+1}\left[1 - \left(\frac{1+q}{2}\right)^{n+1}\right].$$

The fact that the left and right members of (1), when multiplied by $(1 - q)$, are identical establishes the identity (1) for $q \neq 1$.

If $q = 1$, then $s_n = S_n = n + 1$. The left side of (1) becomes

$$\binom{n+1}{1} + \binom{n+1}{2}\cdot 2 + \binom{n+1}{3}\cdot 3 + \cdots + \binom{n+1}{n+1}(n+1) \tag{5}$$

and, from the definition of binomial coefficients

$$\binom{l}{k} = \frac{l!}{k!(l-k)!},$$

(5) may be written

$$\frac{(n+1)!}{n!} + \frac{2(n+1)!}{2!(n-1)!} + \frac{3(n+1)!}{3!(n-2)!} + \cdots + \frac{(n+1)(n+1)!}{(n+1)!}$$

$$= (n+1)\left[1 + \frac{n!}{(n-1)!} + \frac{n!}{2!(n-2)!} + \cdots + \frac{n!}{n!}\right]$$

$$= (n+1)\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right]$$

$$= (n+1)(1+1)^n = (n+1)2^n.$$

But this is precisely the value of $2^n \cdot S_n$ when $q = 1$, so the identity (1) holds also for $q = 1$.

1923/3. Prove that, if the terms of an infinite arithmetic progression of natural numbers are not all equal, they cannot all be primes.

Solution. Let d be the constant difference between consecutive terms a_r and a_{r+1}. Then

$$a_{r+1} = a_r + d \quad \text{and} \quad a_{r+s} = a_r + sd.$$

Since d is a natural number and since $a_1 \geq 1$, all other terms are greater than 1.

Now take any term $a_r > 1$, let $s = a_r$, and consider the sth term after a_r, that is, the term a_{r+s}; it satisfies the relation

$$a_{r+s} = a_r + a_r d = a_r(1 + d),$$

and therefore it is not a prime.

1924 Competition

1924/1. Let a, b, c be fixed natural numbers. Suppose that, for every positive integer n, there is a triangle whose sides have lengths a^n, b^n, and c^n respectively. Prove that these triangles are isosceles.

Solution. Assume that

$$a \geq b \geq c.$$

The numbers a^n, b^n, c^n represent the lengths of the sides of triangles only if $a^n < b^n + c^n$ for all n, i.e., only if

$$a^n - b^n < c^n \qquad \text{for all } n,$$

hence only if

$$(1) \quad (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) < c^n \quad \text{for all } n.$$

Since $c \leq a$ and $c \leq b$, the second factor on the left of (1) is greater than or equal to what it would be if all the a's and b's were replaced by c's. That is,

$$a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1} \geq nc^{n-1}$$

and from (1), the triangles require that

$$c^n > (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}) \geq (a - b)nc^{n-1}$$

or that

$$(2) \qquad (a - b) < \frac{c}{n} \qquad \text{for every } n > 0.$$

But (2) can hold only if $a = b$.

1924/2. If O is a given point, l a given line, and a a given positive number, find the locus of points P for which the sum of the distances from P to O and from P to l is a.

Solution. Denote the distance from P to O by r and that from P to l by ρ; the problem is to determine all points P such that $r + \rho = a$.

If the distance from O to l is greater than a, the problem has no solution. If the distance from O to l is equal to a, the desired locus is the segment OM (Figure 79) from O perpendicular to l.

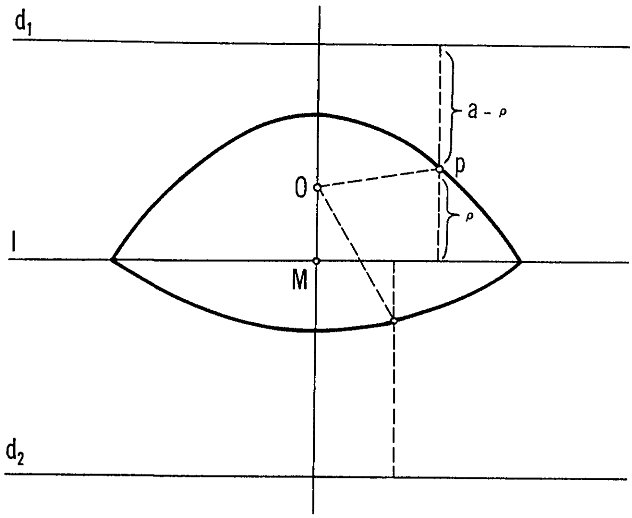

Figure 79

If the distance from O to l is less than a, we find the locus by first drawing two lines, d_1 and d_2, parallel to and at a distance a from l. In the strip between l and d_1, every point P of the desired locus must have a distance ρ from l and the distance $r = a - \rho$ from O; but since the total distance from l to d_1 is a, the distance from P to d_1 is $a - \rho = r$. Therefore, the distance from P to d_1 is equal to the distance from P to O. In other words, every point P in this strip satisfying the conditions of the problem is also a point equidistant from O and d_1, and the locus of all such points consists of a parabola with O as focus and d_1 as directrix. However, only the arc of the parabola

contained in the strip satisfies the conditions of the problem. Similarly, in the strip between l and d_2, the desired locus is the arc of the parabola with focus O and directrix d_2. These two parabolic arcs constitute the desired locus.

1924/3. Let A, B, and C be three given points in the plane; construct three circles, k_1, k_2, and k_3, such that k_2 and k_3 have a common tangent at A, k_3 and k_1 at B, and k_1 and k_2 at C.

Solution. Suppose O_1, O_2, O_3 (see Figure 80) were the centers of the circles satisfying our requirements. The tangents of the pairs of circles meeting in A, B, and C, respectively, are the loci of *points of equal power* with respect to these pairs of circles (see Notes 1 and 2, below); as such they meet in a point O. Moreover, $OA = OB$ (since both are tangent to k_3); similarly, $OB = OC$, so that $OA = OB = OC$. Therefore, O is the center of the circle through A, B, and C. Clearly, the sides of triangle $O_1O_2O_3$, extended if necessary, pass through the points A, B, and C (since they connect the centers of the tangent circles k_1, k_2, and k_3), and are perpendicular to OA, OB, and OC.

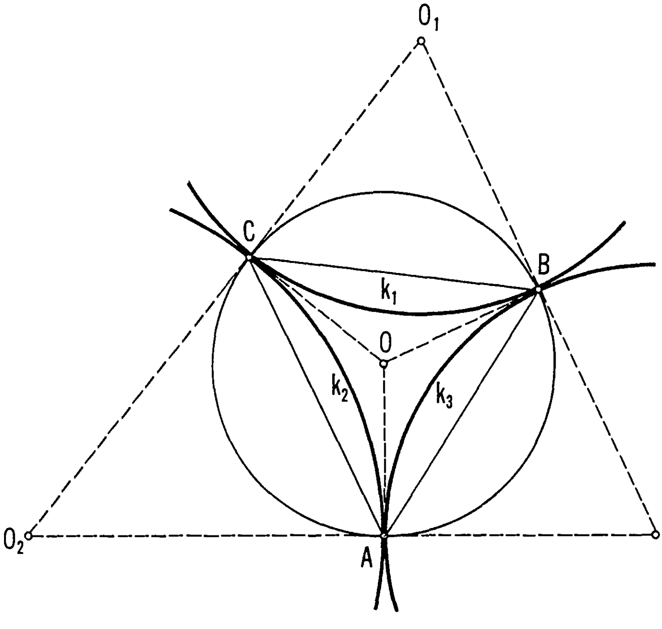

Figure 80

Therefore, the desired circles k_1, k_2, and k_3 are constructed as follows: Triangle ABC is inscribed in a circle; its tangents are drawn at points A, B, and C. The intersection of these tangents will be the centers of k_1, k_2, k_3.

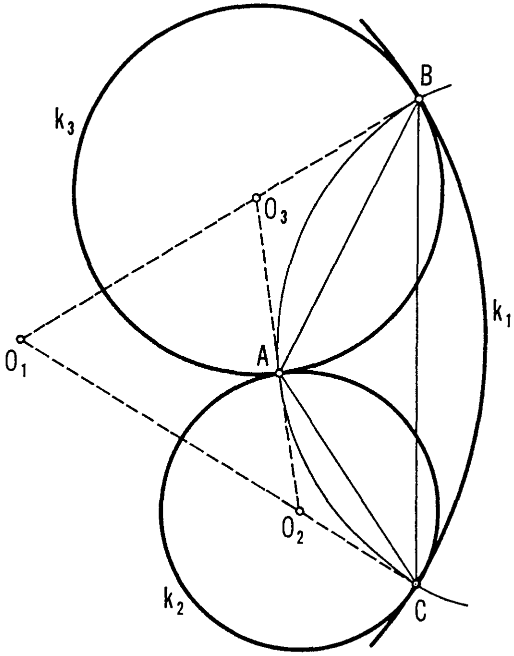

Figure 81

If ABC is an acute triangle, the circles k_1, k_2, k_3 are pairwise tangent externally (Figure 80).

If ABC is an obtuse triangle (Figure 81), then one pair of circles touches externally and both touch the third from inside.

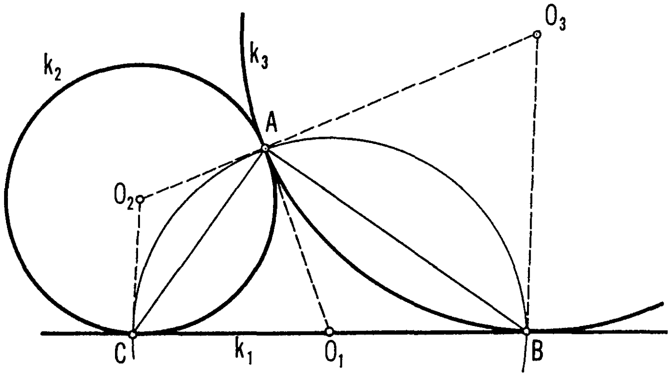

Figure 82

If ABC is a right triangle (Figure 82), two of the three tangents to its circumscribed circle are parallel, so there are only two points of intersection, and the problem has no solution. But we may say that the solution consists of two externally tangent circles, each tangent to a line (that is, a degenerate circle) and on the same side of the line.

Note 1. The power of a point with respect to a circle. It is well known that the product of the distances from a given point P to the points of intersection of a secant through P to a circle k does not depend on the direction of the secant. This can be seen immediately by means of the proportion

$$\frac{PM}{PN'} = \frac{PM'}{PN}$$

of corresponding sides of similar triangles, see Figures 83, 84. In other words

$$PM \cdot PN = \text{constant.}$$

This number is called the *power of point P with respect to the circle k.* PM and PN are viewed as directed line segments; so, if PM and PN have opposite directions (i.e., if P is inside k, see Figure 84), their product is negative; thus, *inner points have negative power*. If P is on the circle k, one of the segments reduces to a point, and so *points on the circle have power zero.*

If P is outside the circle k and the line through P is tangent to k (M and N coalesce), then the power of P is the square of its distance from the point of tangency.

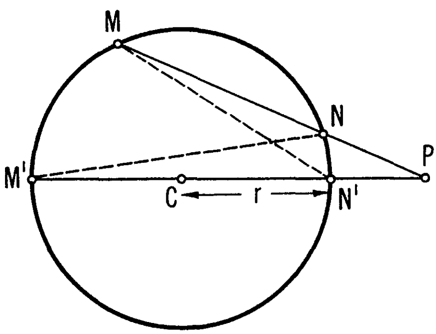

Figure 83

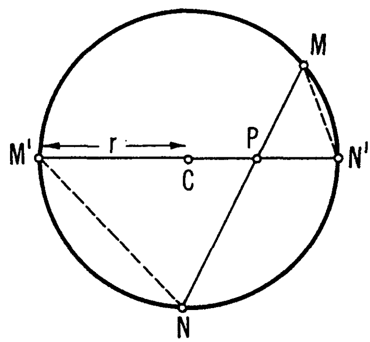

Figure 84

The concept of power of a point combines and generalizes the following familiar theorems: (1) *If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.* (2) *If two secants PAA' and PBB' are drawn to a circle from a point P without, then $PA \cdot PA' = PB \cdot PB'$.* (3) *If a secant PAA' and a tangent PT are drawn to a circle from a point P without, then $PT^2 = PA \cdot PA'$.*

Let r be the radius of k, C its center, and $PC = d$; then if the line through P is a diameter,

$$PM = d + r, \qquad PN = d - r,$$

so that the power of P is

$$(d + r)(d - r) = d^2 - r^2.$$

Note 2. The locus of points whose powers with respect to two circles are equal. *The locus of points in the plane whose powers with respect to two non-concentric circles k_1 and k_2 are the same is a straight line perpendicular to the line through the two centers.*†

PROOF: Let A be the center of k_1, B the center of k_2, P a point whose powers with respect to k_1 and k_2 are being investigated, and Q the foot of the perpendicular from P to AB; see Figure 85. Denote the radii of k_1 and k_2 by a and b, respectively.

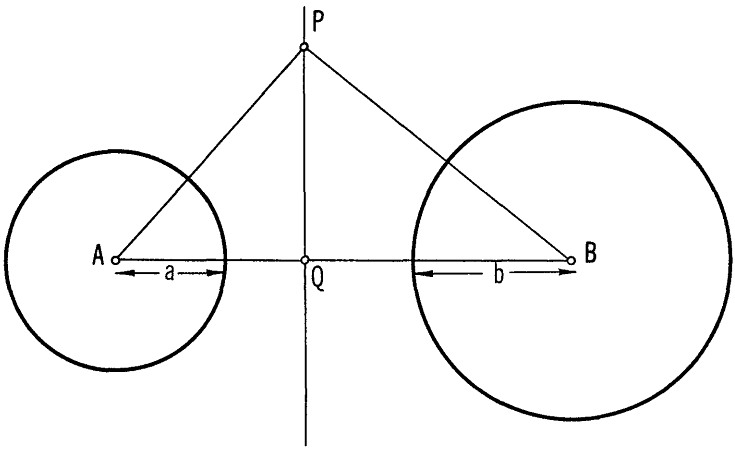

Figure 85

The power of P with respect to k_1 is $(PA)^2 - a^2$, and with respect to k_2, it is $(PB)^2 - b^2$ (see Note 1). If these powers are equal, then

$$(PA)^2 - a^2 = (PB)^2 - b^2, \quad \text{and} \quad b^2 - a^2 = (PB)^2 - (PA)^2.$$

By Pythagoras' theorem,

$$(PB)^2 = (PQ)^2 + (QB)^2, \qquad (PA)^2 = (PQ)^2 + (QA)^2,$$

so that

$$b^2 - a^2 = [(PQ)^2 + (QB)^2] - [(PQ)^2 + (QA)^2] = (QB)^2 - (QA)^2.$$

† This locus is frequently called the *radical axis* of k_1 and k_2.

According to the First Solution of 1912/3, the last difference determines the point Q, and the desired locus is the perpendicular to AB through Q.

If k_1 *and* k_2 *intersect in points* C *and* D, *the locus is the line through* C *and* D. This follows from the observation that the power of C as well as that of D is zero with respect to both circles.

If k_1 *and* k_2 *are tangent at a point* C, *the locus is the common tangent through* C. This follows from the fact that the common tangent contains C and is perpendicular to AB.

If k_1 *lies entirely within* k_2 *and if their centers do not coincide, the locus lies outside* k_2; *but if* k_1 *and* k_2 *are concentric, there is no locus.*

If k_1, k_2, k_3 *are three circles, and each pair has a line of equal powers, then these three lines are either parallel, or they meet in one point.*

The lines will be parallel if the centers of the circles are collinear; for, in this case, the lines are perpendicular to the line through the centers.

If the centers are not collinear, the lines of equal powers are not parallel; the line for k_1 and k_3 meets that for k_2 and k_3 in some point P whose powers with respect to all three circles are the same. Hence P is also on the line of equal powers with respect to k_1 and k_2. This point P is called *the point of equal powers with respect to* k_1, k_2, *and* k_3.

1925 Competition

1925/1. Let a, b, c, d be four integers. Prove that the product of the six differences

$$b-a, \quad c-a, \quad d-a, \quad d-c, \quad d-b, \quad c-b$$

is divisible by 12.

Solution. We shall show that the product P of these differences is divisible by 4 and also by 3. It will follow (cf. 1896/1, 1901/3, Notes) that P is divisible by 12.

a) We classify the given integers a, b, c, d according to the remainders they have upon division by 4; since the only remainders are 0, 1, 2, 3, there are only 4 possible classes.

If two of the given integers fall into the same class, that is, have the same remainder upon division by 4, then their difference is divisible by 4. Since the factors in P include all possible differences (except for sign) of the given numbers, one of the factors and hence P will be divisible by 4.

If no two of the given integers have the same remainder upon division by 4, then they have remainders 0, 1, 2, and 3, respectively, so that two

of the numbers are even and two odd. The difference of the even ones as well as that of the odd ones is even, so P is divisible by 4.

b) We next classify the four given integers according to their remainders upon division by 3. Since there are only the three possible remainders 0, 1, 2, at least two of the numbers must fall into the same class (cf. 1906/3, Note) and the difference of these two is divisible by 3. Hence P is also divisible by 3.

1925/2. How many zeros are there at the end of the number

$$1000! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 999 \cdot 1000\ ?$$

Solution. The number of terminal zeros of a number depends on how often the factor $10 = 2 \cdot 5$ occurs in its factorization. We must therefore find the exponents of the factors 2 and 5 in the prime factorization of 1000! (cf. 1896/1, 1901/3, Notes). The smaller of these exponents will yield the largest exponent, say α, for which $10^\alpha = (2 \cdot 5)^\alpha$ divides 1000!, and α will be the number of terminal zeros in 1000!.

Now, every fifth one of the numbers

$$1, \quad 2, \quad 3, \quad \cdots, \quad 1000$$

is a multiple of 5. Since

$$1000 = 5 \cdot 200 + 0,$$

there are 200 factors in 1000! which are divisible by 5. Of these 200, i.e., of

$$5, \quad 10, \quad 15, \quad 20, \quad 25, \quad \cdots, \quad 1000,$$

every fifth is a multiple of 5^2. Since

$$200 = 5 \cdot 40 + 0,$$

there are 40 factors divisible by 5^2. Moreover, since

$$40 = 5 \cdot 8 + 0, \qquad 8 = 5 \cdot 1 + 3, \qquad 1 = 5 \cdot 0 + 1,$$

there are 8 numbers divisible by 5^3, one by 5^4, and none by any higher power of 5.

Thus, the prime factorization of 1000! contains $5^{200} \cdot 5^{40} \cdot 5^8 \cdot 5^1$; the exponent of 5 in the prime factorization is

$$200 + 40 + 8 + 1 = 249.$$

On the other hand, the prime 2 occurs to a much higher power in the prime factorization of 1000!, since 500 of the factors are even, 250 divisible by 2^2, etc. Hence there are 249 zeros at the end of 1000!.

Note. The exponents in the prime factorization of m!. Let us consider the general problem: What is the highest power, say p^α, of a prime p such that p^α divides $m!$, where m is any given natural number? In order to answer this question, we shall generalize the procedure used in the above solution for the case $m = 1000$, $p = 5$.

We divide the given number m by the prime p:

$$m = pq_1 + r_1 \qquad (0 \leq r_1 < q_1);$$

next, we divide the quotient q_1 by p:

$$q_1 = pq_2 + r_2 \qquad (0 \leq r_2 < q_2).$$

We continue this process until we obtain a quotient which is zero, i.e.,

$$q_{k-1} < p, \qquad q_k = 0.$$

By examining the solution of the problem, we see that the exponent α (which was 249 in the case $m = 1000$, $p = 5$) is the sum of the quotients obtained in the above algorithm:

$$\alpha = q_1 + q_2 + \cdots + q_{k-1}.$$

We now claim that the remainders $r_1, r_2, \cdots, r_k$ obtained in this algorithm are the digits in the representation of the number m to the base p. To see this, we just substitute repeatedly for the quotients; thus,

$$m = pq_1 + r_1 = p(pq_2 + r_2) + r_1 = \cdots = p^{k-1}r_k + p^{k-2}r_{k-1} + \cdots + pr_2 + r_1.$$

The last expression proves our claim. (In the solution of the problem, the remainders in the divisions were 0, 0, 0, 3, 1 and the representation of 1000 to the base 5 is, indeed, 13000.)

Next we shall show how to express the exponent α in terms of the remainders. We add the expressions

$$\begin{aligned} m &= pq_1 + r_1 \\ q_1 &= pq_2 + r_2 \\ &\vdots \\ q_{k-1} &= p \cdot 0 + r_k \end{aligned}$$

and obtain

$$m + q_1 + q_2 + \cdots + q_{k-1} = p(q_1 + q_2 + \cdots + q_{k-1}) + r_1 + r_2 + \cdots + r_k$$

or

$$m + \alpha = p\alpha + s,$$

where $s = r_1 + r_2 + \cdots + r_k$. Solved for α, this gives

$$\alpha = \frac{m - s}{p - 1}.$$

This proves the following theorem of Legendre:†

If m is a positive number and p a prime, then the exponent of p in the prime factorization of $m!$ is

$$\frac{m - s}{p - 1}$$

where s is the sum of the digits of the representation of m to the base p.

1925/3. Let r be the radius of the inscribed circle of a right triangle ABC. Show that r is less than half of either leg and less than one fourth of the hypotenuse.

Solution. Let $a = BC$, $b = CA$ be the legs and $c = AB$ the hypotenuse of $\triangle ABC$; let D be the foot of the altitude from C. Denote by E, F, G the points of tangency of the inscribed circle with a, b, c, and by E', F', G' the points on the circle diametrically opposite E, F, G (Figure 86).

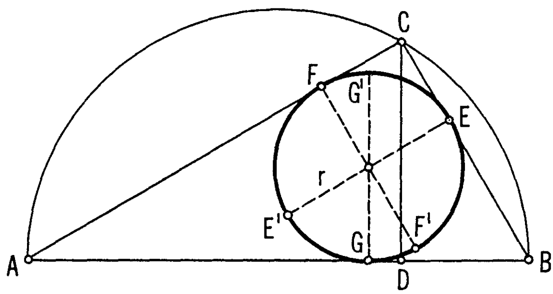

Figure 86

Of all points on and inside a triangle, the one *farthest* from a particular side of the triangle is the vertex opposite that side. For example, among all points in or on $\triangle ABC$, A is farthest from BC. Therefore

(1) $$E'E < AC, \qquad F'F < BC, \qquad G'G < CD.$$

† A. M. Legendre (1752–1833) was the director of the University of Paris and a member of the Academy of Science. His works on number theory are: *Essai sur la théorie des nombres* (1798), and *Théorie des nombres* (1830).

Now, since CD is half of a chord of the circle with diameter AB, $CD \leq \frac{1}{2}AB$, so that

$$G'G < \tfrac{1}{2}AB. \tag{2}$$

From the inequalities (1) and (2), it follows that

$$2r < b, \qquad 2r < a, \qquad 2r < \tfrac{1}{2}c; \tag{3}$$

that is, r is less than $\frac{1}{2}a$, it is less than $\frac{1}{2}b$, and it is less than $\frac{1}{4}c$.†

1926 Competition

1926/1. Prove that, if a and b are given integers, the system of equations

$$x + y + 2z + 2t = a$$

$$2x - 2y + z - t = b$$

has a solution in integers x, y, z, t.

Solution. Suppose that the given system of equations had integer solutions for the special values $a = 1$, $b = 0$, and also for the special values $a = 0$, $b = 1$; i.e., suppose that integers x_1, y_1, z_1, t_1 solved the system

$$(1) \qquad \begin{aligned} x + y + 2z + 2t &= 1 \\ 2x - 2y + z - t &= 0, \end{aligned}$$

and that the integers x_2, y_2, z_2, t_2 solved the system

$$(2) \qquad \begin{aligned} x + y + 2z + 2t &= 0 \\ 2x - 2y + z - t &= 1. \end{aligned}$$

† The first two inequalities in (1) also follow from the relation

$$a + b = c + 2r.$$

To establish it, write

$$a = BE + EC = BE + r, \qquad b = AF + FC = AF + r$$

and

$$a + b = AF + BE + 2r.$$

Since $AF = AG$, $BE = BG$, and $AG + BG = c$, the equality follows.

Then the given system

$$(3)\qquad \begin{aligned} x + y + 2z + 2t &= a \\ 2x - 2y + z - t &= b \end{aligned}$$

would have the solution

$$(4)\qquad x = ax_1 + bx_2, \quad y = ay_1 + by_2, \quad z = az_1 + bz_2, \quad t = at_1 + bt_2.$$

This may be verified by direct substitution:

$$\begin{aligned} x + y + 2z + 2t &= a(x_1 + y_1 + 2z_1 + 2t_1) + b(x_2 + y_2 + 2z_2 + 2t_2) \\ &= a \cdot 1 + b \cdot 0 = a \\ 2x - 2y + z - t &= a(2x_1 - 2y_1 + z_1 - t_1) + b(2x_2 - 2y_2 + z_2 - t_2) \\ &= a \cdot 0 + b \cdot 1 = b, \end{aligned}$$

and the expressions (4) are clearly integers.

In order to find a special solution x_1, y_1, z_1, t_1 of (1), we observe that the first equation requires that $x_1 + y_1$ be odd. If we set $x_1 = 1$, $y_1 = 0$, we see that $z_1 = -1$, $t_1 = 1$ solve (1).

For system (2), $z_2 - t_2$ must be odd; the integers $x_2 = -1$, $y_2 = -1$, $z_2 = 1$, $t_2 = 0$ solve (2). It follows from (4) that the integers

$$x = a - b, \quad y = -b, \quad z = -a + b, \quad t = a$$

constitute a solution of the given system (3).

Note. Moves of the knight on an infinite chessboard. The theorem we have proved is equivalent to the statement: *On an infinite chessboard, any square can be reached by the knight in a sequence of appropriate moves.*

An infinite chessboard differs from the usual (8×8) one in that it extends over the entire plane. Pick, as the origin of a rectangular coordinate system, the center of one of these infinitely many squares of the chessboard, and let the coordinate axes run parallel to the sides of the squares; see Figure 87. If our unit of length is the side of a square, then the centers of all the squares will have integer coordinates, i.e., will be *lattice points* in our coordinate system.

Now, the result of a succession of moves on a chessboard may be characterized by the coordinates of the beginning lattice point of the moves and those of the end lattice point, that is, by the difference in the coordinates of the endpoints of the path of the figure. In particular, the eight distinct moves of a knight (see Figure 87) are described by the pairs of integers

$$u_1=(\ \ 1,\ \ 2),\quad u_2=(\ \ 1,-2),\quad u_3=(\ \ 2,\ \ 1),\quad u_4=(\ \ 2,-1),$$
$$-u_1=(-1,-2),\quad -u_2=(-1,\ \ 2),\quad -u_3=(-2,-1),\quad -u_4=(-2,\ \ 1).$$

The moves u_i and $-u_i$ are opposite in the sense that they cancel each other; for example, the move u_1 made seven times followed by the move $-u_1$ five times has the net effect of the move u_1 made twice. If we make the move u_1 eight times and the move $-u_1$ twelve times, we may say that we have made the move u_1 (-4) times.

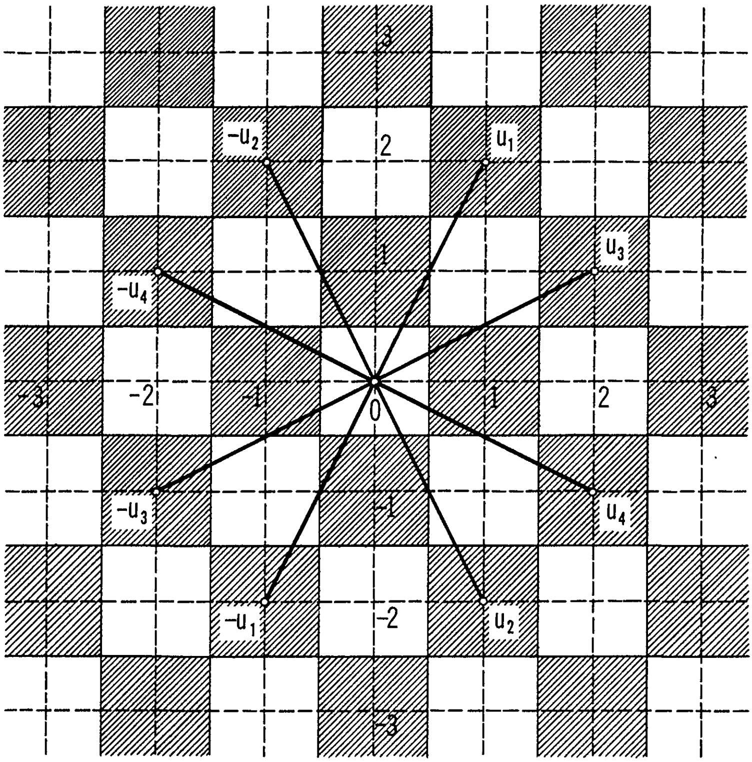

Figure 87

With this agreement, the move u_1 executed x times leads to the change in coordinates $(x, 2x)$, and the move u_2 executed y times leads to $(y, -2y)$; similarly, z times the move u_3 gives $(2z, z)$ and t times u_4 gives $(2t, -t)$. Clearly the result of all these moves together is represented by

$$(5) \qquad (x + y + 2z + 2t,\quad 2x - 2y + z - t).$$

Now, a given square (a, b) can be reached from the origin by a knight if x, y, z, t can be so chosen that (5) yields (a, b), i.e., if x, y, z, t satisfy the system of equations (3) given in the problem.

Having established the equivalence of the problem posed originally with this chessboard problem, we can now give a short proof: Figure 88 shows how the knight can reach the square immediately to the right of the starting square in three moves. Similarly, the knight can go from $(0, 0)$ to $(0, 1)$, or from $(0, 0)$ to $(-1, 0)$, or from $(0, 0)$ to $(0, -1)$. In other words, the knight can reach each of the four adjacent squares. Since the path from $(0, 0)$ to any square (a, b) can be thought of as a sequence of horizontal and vertical steps (one square at a time), the knight can certainly reach any specified square. In fact, the knight can accomplish this in at most $3(a + b)$ moves.

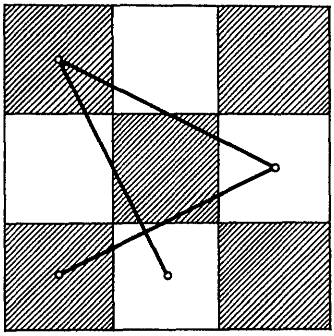

Figure 88

1926/2. Prove that the product of four consecutive natural numbers cannot be the square of an integer.

Solution. The product of n, $n + 1$, $n + 2$, and $n + 3$ is

$$\begin{aligned}[n(n+3)][(n+1)(n+2)] &= (n^2+3n)(n^2+3n+2) \\ &= (n^2+3n)^2 + 2(n^2+3n) \\ &= [(n^2+3n)^2+2(n^2+3n)+1] - 1 \\ &= (n^2+3n+1)^2 - 1.\end{aligned}$$

The second line in these equalities tells us that this product is greater than the square of $n^2 + 3n$ and the last line that it is less than the square of the next integer, $n^2 + 3n + 1$; and so it is *between* the squares of the consecutive numbers

$$n^2 + 3n \quad \text{and} \quad n^2 + 3n + 1.$$

1926/3. The circle k' rolls along the inside of circle k; the radius of k is twice the radius of k'. Describe the path of a point on k'.

Solution. Suppose that, initially, the point M of the smaller circle coincides with the point A of the larger circle (see Figure 89); we shall follow the path of M as k' rolls along k. Observe that at all times, the radius OB of k to a point B of tangency is a diameter of k'. Also, the arc BM of k' has the same length as the arc AB of k. Since the radius of k' is half the radius of k, the central angle supported by arc BM is double the central angle BOA of k. Therefore, the angle BOM inscribed in k' is equal to the central angle BOA of k. It follows that M is on the radius OA of k.

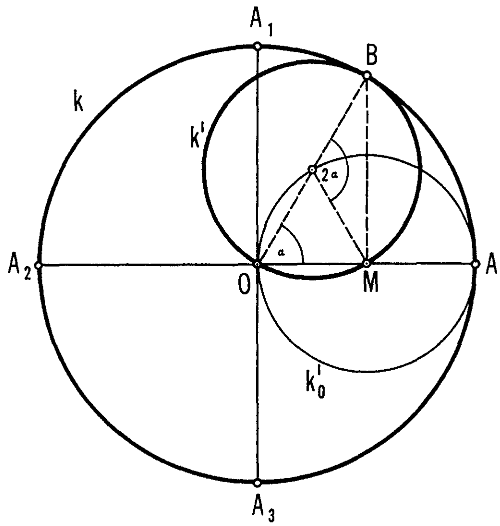

Figure 89

Since OB is a diameter of k, the angle OMB is a right angle; M is the foot of the perpendicular from B to OA. As k' rolls from A to A_1, M moves from A to O along the radius AO of k; as k' continues to move from A_1 to A_2, the path of M is the reflection (in A_1A_3) of the previous path but traversed in the opposite order. That is, as k' rolls from A_1 to A_2, M moves from O to A_2 along OA_2. As k' continues to roll from A_2 to A_3 back to A, M moves from A_2 to A along the diameter A_2A of k, and if k' continues to roll, M repeats this path.

1927 Competition

1927/1. Let the integers a, b, c, d be relatively prime to

$$m = ad - bc.$$

Prove that the pairs of integers (x, y) for which $ax + by$ is a multiple of m are identical with those for which $cx + dy$ is a multiple of m.

First Solution. First we note that a and b are relatively prime; for, if they had a common divisor greater than 1, this divisor would divide

$$m = ad - bc$$

and m would not be prime to a and b, contrary to the hypothesis.

Let (x, y) be a pair of integers for which $ax + by = mk$, where k is an integer. Then

$$ax + by = k(ad - bc),$$

that is

$$a(x - kd) = -b(y + kc) \tag{1}$$

so that the product of b and $y + kc$ is a multiple of a. Since a is prime to b, $y + kc$ must be a multiple of a [cf. 1901/3, Note 3, (ii)], i.e.,

$$y + kc = la, \qquad l \text{ an integer.}$$

If we substitute this in (1) and divide by a, we obtain

$$x - kd = -lb$$

so that

$$x = kd - lb, \qquad y = -kc + la. \tag{2}$$

But then

$$cx + dy = l(ad - bc) = lm$$

which shows that $cx + dy$ is a multiple of m. Thus: *if x and y are integers for which $ax + by$ is a multiple of m, then $cx + dy$ is also a multiple of m.*

Second Solution. If we set $u = ax + by$, $v = cx + dy$, then

$$du - bv = (ad - bc)x = mx$$

or

$$bv = du - mx. \tag{3}$$

If x, y are integers for which u is a multiple of m then, in view of (3), the same holds for bv; and since b is relatively prime to m, v is a multiple of m.

Similar reasoning shows that, if v is a multiple of m, so is u.

For $m = 17$, $u = 2x + 3y$, $v = 9x + 5y$, this yields the Second Solution of 1894/1.

1927/2. Find the sum of all distinct four-digit numbers that contain only the digits 1, 2, 3, 4, 5, each at most once.

Solution. Consider a fixed place P (say the unit's digit) of a four-digit number, and consider a fixed number n among the numbers 1, 2, 3, 4, 5. In the set of all four-digit numbers under consideration, n occurs in the Pth place as often as the number of ways in which the remaining three places can be filled with three of the remaining four numbers. This can be done in $4 \cdot 3 \cdot 2 = 24$ ways.

Then, the sum of the digits in the Pth places of all our numbers is

$$24(1 + 2 + 3 + 4 + 5) = 24 \cdot 15 = 360,$$

and so the sum of the numbers themselves is

$$360(10^3 + 10^2 + 10 + 1) = 360 \cdot 1111 = 399,960.$$

1927/3. Consider the four circles tangent to all three lines containing the sides of a triangle ABC; let k and k_c be those tangent to side AB between A and B. Prove that the geometric mean of the radii of k and k_c does not exceed half the length of AB.

First Solution. Let $AB = c$, let r be the radius of the inscribed circle k, and let r_c be the radius of the excircle k_c. We established the necessary and sufficient condition

$$r \cdot r_c \leq \frac{c^2}{4}$$

for the existence of a solution to 1900/2. This relation proves also the present statement, since it is equivalent to the desired inequality

$$\sqrt{r \cdot r_c} \leq \frac{c}{2}.$$

Second Solution. Let T be the area of triangle ABC, and let

$s = \frac{1}{2}(a + b + c)$ be half its perimeter. According to (4) and (7) of 1896/3, Note 2,

$$T = rs = r_c(s - c)$$

whence

$$r = \frac{T}{s}, \qquad r_c = \frac{T}{s - c},$$

so that

$$r \cdot r_c = \frac{T^2}{s(s - c)} = \frac{s(s - a)(s - b)(s - c)}{s(s - c)} = (s - a)(s - b),$$

or

$$\sqrt{r \cdot r_c} = \sqrt{(s - a)(s - b)}.$$

Since the geometric mean of two numbers never exceeds their arithmetic mean (cf. 1916/2, Note 1), we conclude that

$$\sqrt{r \cdot r_c} \leq \frac{(s - a) + (s - b)}{2} = \frac{\frac{1}{2}(b + c - a) + \frac{1}{2}(a + c - b)}{2} = \frac{c}{2}.$$

Third Solution.† Our starting point is the following statement (see Figure 90a): *The vertices A and B and the centers O and O' of k and k_c lie on a circle, and AB separates O and O'.*

To see this note that angles OAO' and OBO' are right angles since bisectors of adjacent supplementary angles are perpendicular to each other. Thus OO' is a diameter of the circle on which also A and B lie. Moreover, since AB separates k and k_c, it certainly separates their centers O and O'.

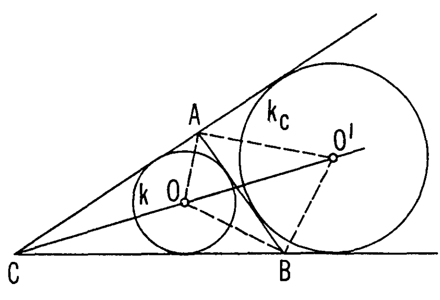

Figure 90a

† This solution is due to Lipót Fejér.

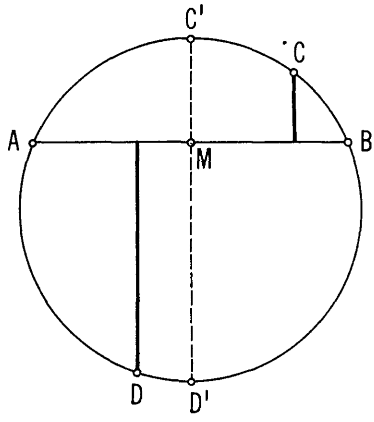

Figure 90b

Our statement reduces the problem to a special case of the following theorem (Figure 90b):

If the points A, B separate the points C, D on a circle, then the geometric mean of the distances from C to AB and from D to AB is at most $\frac{1}{2}AB$. These distances are greatest when C, D are endpoints of the diameter $C'D'$ perpendicular to AB. In this case the geometric mean is exactly $\frac{1}{2}AB$ since

$$MC' \cdot MD' = MA \cdot MB = \tfrac{1}{4}(AB)^2.$$

1928 Competition

1928/1. Prove that, among the positive numbers

$$a, 2a, \cdots, (n-1)a,$$

there is one that differs from an integer by at most $1/n$.

Solution. The problem states a special case of the theorem of Dirichlet–Kronecker† which may be proved by the box principle (cf.

† L. Dirichlet (1805–1859) was professor first in Berlin and then (as Gauss' successor) in Göttingen. His best-known theorem is:

If a and d are relatively prime, then the infinite arithmetic progression

$$a, a + d, a + 2d, \cdots$$

contains infinitely many primes.

L. Kronecker (1823–1891) was professor in Berlin.

1906/3, Note). Roughly, it states that, for large n, at least one of these numbers is "almost" an integer. The theorem states exactly how to interpret this "almost".

We use the device shown in Figure 91 for the case $n = 12$; that is, we draw a circle, use its circumference as the unit of length, and mark off the $n - 1$ lengths $a, 2a, \cdots, (n-1)a$ along the circumference, beginning at some point O. (A circle of circumference 1 is convenient because we are interested only in the fractional portions of our numbers. For example, $e = 2.717\cdots$ and $.717\cdots$ are equivalent in this problem.)

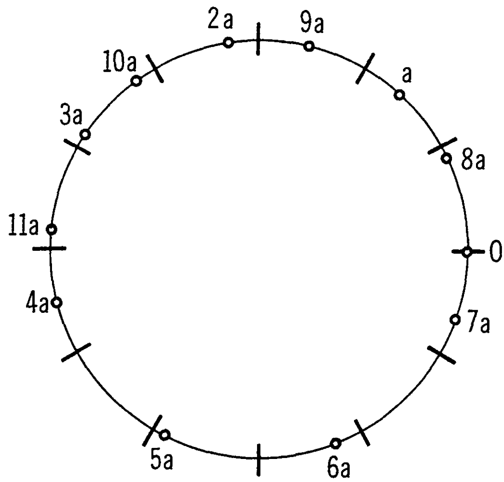

Figure 91

Next, we divide the circle into n equal arcs of length $1/n$, starting at the point O; we consider a division point as belonging only to the arc of which it is the endpoint. In this way, every point on the circle belongs to exactly one arc.

Our object is to show that one of the arcs adjacent to point O necessarily contains one of our $n - 1$ marked points. If this were not so, then all $n - 1$ marked points would belong to the $n - 2$ remaining arcs and hence, by the box principle (see 1906/3, Note), one of these $n - 2$ arcs must contain two of the marked points. In the figure the points $3a$ and $10a$ are on the same arc, and so their difference is less than $1/n$. But then the distance between the division point at O and the marked point

$$10a - 3a = 7a$$

is also less than $1/n = 1/12$ which shows, contrary to our assumption, that one of the arcs adjacent to O must contain one of the $n - 1$ points. In this case, $7a$ differs from an integer by at most $1/12$.

We used the illustrative example $n = 12$ and spoke of the numbers $11a$, $3a$, $10a$, and $7a$ instead of the corresponding numbers $(n-1)a$, ma, ka, and $(k-m)a$ that figure in the general case.

1928/2. Put the numbers $1, 2, 3, \cdots, n$ on the circumference of a circle in such a way that the difference between neighboring numbers is at most 2. Prove that there is just one solution (if regard is paid only to the order in which the numbers are arranged).

Solution. The neighbors of the number 1 can only be 2 and 3 (Figures 92, 93). The other neighbor of 2 must then be 4, and the other neighbor of 3 must be 5. Similarly the next neighbor of 4 must be 6, etc., so that we have the succession $2, 4, \cdots$ of even numbers on one side of 1 and the succession $3, 5, \cdots$ of odd numbers on the other side of 1; the neighbors of n will be $n-2$ and $n-1$. Clearly this is a solution and the only possible one.

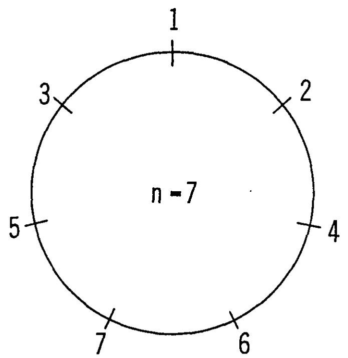

Figure 92

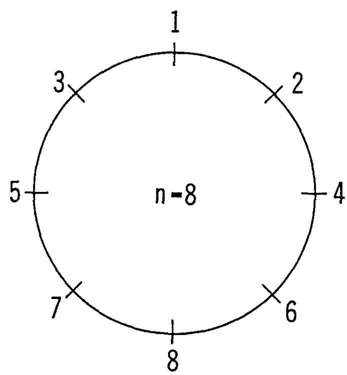

Figure 93

1928/3. Let l be a given line, A and B given points of the plane. Choose a point P on l so that the longer of the segments AP, BP is as short as possible. (If $AP = BP$, either segment may be taken as the "longer" one.)

Solution. Suppose A is no closer to l than B is; denote by A_1 the foot of the perpendicular from A to l (Figure 94).

1. If $AA_1 \geq A_1B$, A_1 is the desired point P; AA_1 is the "longer" segment, and for every other point Q on l, $AQ > AA_1$.

2. If $AA_1 < A_1B$, draw the perpendicular bisector f of the segment AB and let B_1 be the foot of the perpendicular from B to l (Figure 95).

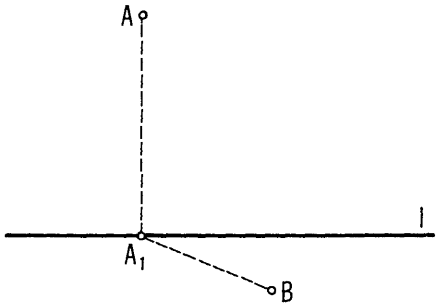

Figure 94

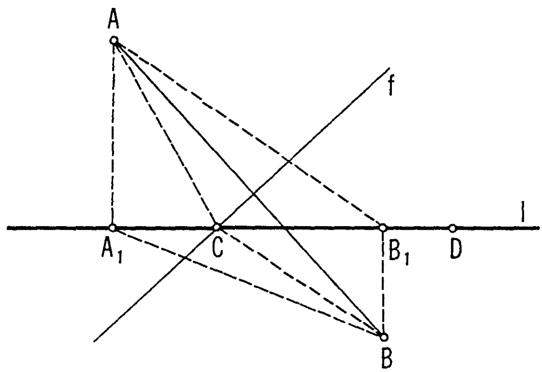

Figure 95

Any point in the plane that is closer to A than to B is on the same side of f as A is, and any point closer to B than to A is on the same side of f as B. Now A_1 is on the same side of f as A, for $A_1A < A_1B$; moreover B_1 is on the same side of f as B because $B_1B \leq A_1A$ (we supposed that A is no closer to l than B is), and $A_1A < B_1A$ (A_1 is the foot of the perpendicular from A), so that $B_1B < B_1A$. Hence f intersects the segment A_1B_1 in a point C.

To see that C is the desired point in this case, note that $AC = BC$, while for any other point Q on l, either AQ or BQ exceeds AC. For example, choose a point D on l, on the same side of f as B_1; then $AD > AC$ because AD is the side opposite the obtuse angle in triangle ACD. Hence, the longer of the segments AD, BD certainly exceeds AC. A similar argument can be used to exclude a point D' on l, on the other side of f.

Classification† of Problems

Number Theory:	1906/3,	1907/1,	1907/3,
	1908/1,	1909/1,	1910/2,
	1911/3,	1912/2,	1913/3,
	1917/1,	1917/2,	1922/2,
	1923/3,	1925/1,	1925/2,
	1926/2,	1927/1.	
Combinatorics:	1912/1,	1916/3,	1927/2,
	1928/2.		
Quadratic equations:	1916/1,	1917/1.	
Quadratic functions:	1914/2,	1918/3.	
Diophantine equations:	1918/2,	1926/1.	
Number sequences:	1922/3,	1923/2,	1923/3.
Inequalities:	1910/1,	1911/1,	1913/1,
	1914/2,	1915/1,	1916/1,
	1918/3,	1922/3,	1928/1.

† Since the problems and specially their solutions often involve several mathematical disciplines, this classification is necessarily arbitrary and somewhat incomplete.

Plane geometry (without trigonometry)

Proofs:	1906/2, 1907/2, 1908/3, 1909/3, 1912/3, 1914/1, 1914/3, 1917/3, 1918/1, 1923/1, 1924/1.
Computations:	1910/3, 1911/2.
Loci:	1924/2, 1926/3.
Constructions:	1924/3.
Extreme values:	1928/3.
Inequalities:	1907/2, 1908/2, 1909/2, 1914/1, 1915/2, 1915/3, 1916/2, 1925/3, 1927/3.

Angle measurement:	1906/1.
Trigonometry:	1906/1, 1909/2, 1911/2.
Solid geometry:	1913/2, 1922/1.
Applications of the box principle:	1906/3, 1907/3, 1925/1, 1928/1.

List of Explanatory Notes

List of Biographical Notes

List of Winners

1906	Vilmos Erdős		István Gotléb
1907	Jenő Tolnai		György Domokos
1908	Etelka Orphanides		Lajos Kudlák
1909	László Raj		Ferenc Lukács
1910	Ödön Sebestyén		
1911	Károly Hlucsil		Gábor Klein
1912	Gábor Szegő		Pál Neményi
1913	Tibor Radó		Lajos Fülep
1914	Ferenc Zigány		Jenő Prónai
1915	Alfréd Boskovitz		Ferenc Krbek
1916	Albert Kornfeld		Kálmán Hajnal
1917	Celesztin Z. Pomázi		Lajos Sárospataky
1918	Endre Reuss		László Rédei
1922	László Kalmár		Vilmos Schmidt
1923	György Ländler		Miklós Izsák
1924	Endre Körner		
1925	Rudolf Fuchs	Ede Teller	László Tisza
1926	Tibor Bakos		József Winkler
1927	Gyula Beke		Miklós Ság
1928	György Schossberger		Endre Schlüssler

The Eötvös Contests in elementary mathematics have been open to Hungarian students in their last year of high school ever since 1894. They are famous for the simplicity of the concepts employed, the mathematical depth reached, and the diversity of elementary mathematical fields touched. But perhaps their most remarkable feature is the influence that they, together with a mathematics journal for students, seem to have had on the young people of that small country. Among the winners of the first eleven contests (i.e. those contained in NML volume 11), many turned into scientists of international fame; e.g. L. Fejér, T. von Kármán, D. Kőnig, M. Riesz. Among the winners of the next twenty contests (i.e. those contained in the present volume) are G. Szegő, T. Radó, E. Teller; all three are well-known in the United States, where they now reside. This translation of the Eötvös Contest Problems from 1894-1928 is based on the revised Hungarian edition of J. Kürschák's original compilation. Kürschák combined his excellence in mathematics with his interest in education when he supplied the elegant solutions and illuminating explanations.

JÓZSEF KÜRSCHÁK (1864-1933) was born and educated in Hungary. He was professor of mathematics at the Polytechnic University in Budapest, member of the Hungarian Academy and permanent member of the Examination Board for prospective high school teachers of mathematics.

His many contributions to mathematics include work in the calculus of variations, in algebra and in number theory. He used his great pedagogical skill in developing and teaching an exceptionally good mathematics course for beginning engineering students. He also gave courses for future high school teachers, mainly in elementary geometry and in geometrical constructions. Several of his papers deal with the teaching and popularization of mathematics. His devotion to intelligently guided problem solving is illustrated by the famous problem book which forms the basis of the present volume.